Real Analysis I

Acknowledgements

These notes follow closely from Chapters 1-7 Rudin's *Principles*. They are based on classes I have taken from Logan Stokols (Duke), David Cruz-Uribe, and Tim Ferguson (both UA). In particular, the topics not covered in Rudin and the homework problems at the end are from Logan's class. I would also like to emphasize DCU's notes as being particularly helpful and comprehensive. All these instructors were incredible and personable in their approach/interactions; I can't thank them enough. As far as the format of these notes, there is little extraneous discussion or commentary. These are mostly definitions and results stated in a linear fashion. There is a lot of shorthand used, some of which is defined after the notes. This was crafted with only myself in mind but may be helpful to others. Also: I corrected the HW problems I was marked off for but cannot guarantee there are no mistakes

1 Real and Complex number systems

lots of information skipped for this chapter

A **totally ordered** set is a set S with a *total order* $<, >, \leq, \geq$ s.t $\forall\, x, y, z \in S$ either $x < y$ or $x > y$ or $x = y$ and (transitivity) if $x < y, y < z \implies x < z$.
An **ordered field** is a *field* F (satisfies usual addition and multiplication axioms, see Rudin) that is an ordered set and $x, y, z \in F, y < z \implies x + y < x + z$ and $x, y \in F, x, y > 0 \implies xy > 0$.

Theorem (Cauchy-Schwarz Inequality) For any ordered field F, $\forall\, x, y \in F\;\; xy \leq .5(x^2 + y^2)$

An ordered set S is **(Dedekind) complete**, or has the *least upper bound property*, if $\forall E \subseteq S$ bounded above and nonempty $\exists \alpha \in S$ s.t $\alpha = \sup E$.

Theorem For $x \in \mathbb{R}$ and $n \in \mathbb{N}$ $\exists\, y \in \mathbb{R}$ s.t $y = x^n$
Proof: This is a proof for $\exists y \in \mathbb{R}^+$ s.t $y^3 = 3$, which can be adapted for the general case by using different constraints (i.e. w.r.t x and n instead of 3). Define the set $E = \{x \in \mathbb{R}^+ | x^3 < 3\}$. We know $\sup E$ exists by the (Dedekind) completeness of $\mathbb{R}$. We want to show that $(\sup E)^3 = 3$.
Case 1 Define arbitrary $\alpha > 1$ s.t $\alpha^3 < 3$. We will show α isn't an upper bound. Let $\delta = 3 - \alpha^3$, implying $\delta \in (0, 1)$. Fix $\epsilon > 0$ s.t $\epsilon < \delta/(9\alpha^2) < \delta/9$. Note that for $n \in \mathbb{N}\backslash\{1\}$ $\alpha^n > \alpha$ and $\epsilon^n < \epsilon$. Then

$$(\alpha + \epsilon)^3 = \alpha^3 + \epsilon^3 + 3(\alpha\epsilon^2 + \alpha^2\epsilon) < \alpha^3 + \epsilon(1 + 6\alpha^2) < \alpha^3 + (7\delta)/9 < \alpha^3 + \delta = 3$$

Therefore, $\alpha + \epsilon$ is not an upper bound of E, so neither is α
Case 2 Now define $\alpha \in (1, 2)$ s.t $\alpha^3 > 3$. We will show α is not the least upper bound of E. Let $\delta = \alpha^3 - 3$, so $\delta \in (0, 1)$. Fix $\epsilon > 0$ s.t $\epsilon < \delta/(6\alpha^2) < \delta/6$. Since $\epsilon^3 > 0$, $-\epsilon^2 > -\epsilon$, and $-\alpha > -\alpha^2$

$$(\alpha - \epsilon)^3 = \alpha^3 + \epsilon^3 - 3(\alpha\epsilon^2 + \alpha^2\epsilon) > \alpha^3 - 6\epsilon\alpha^2 > \alpha^3 - \delta = 3$$

Therefore, α can't be the least upper bound because $\alpha - \epsilon$ is an upper bound
As mentioned, we know $\sup E$ exists. Further, $\sup E \in \mathbb{R}^+$ from the cases above. Define $y = \sup E$. In both cases, we defined α arbitrarily, meaning that we can make the general statement that for $u \in \mathbb{R}$, if $u^3 < 3$ or $u^3 > 3$, then $u \neq \sup E$. By contraposition, $y^3 = (\sup E)^3 = 3$ $\blacksquare$.

Theorem (Density of $\mathbb{Q}$) $\forall x, y \in \mathbb{R}\; x > y \implies \exists\, p \in \mathbb{Q}$ s.t $x > p > y$

Theorem (Archemedian Property) $x, y \in \mathbb{R}^+ \implies \exists\, n \in \mathbb{N}$ s.t $nx > y$. Also, $\exists\, m \in \mathbb{Q}$ s.t $m > x > m^{-1}$.

Dot Product $x \cdot y = \sum_{i=1}^{n} x_i y_i$
For $n \in \mathbb{N}$, the n-dimensional **Euclidean Space** is $\mathbb{R}^n$ (n-tuple) with dot product and norm $||x|| = \sqrt{x \cdot x}$

Theorem (Cauchy-Schwarz) $x, y \in \mathbb{R}^n \implies x \cdot y \leq ||x|| \cdot ||y||$
Proof: $0 \leq ||x + ty||^2 = ||x + ty||^2 = x \cdot x + 2t(x \cdot y) + t^2(y \cdot y) = ||x||^2 + (2t)x \cdot y + (t||y||)^2$
The last term is a non-negative quadratic ($\forall t$), so all $D \leq 0$ (discriminants, e.g. $4(x \cdot y)^2 - 4||x||^2||y||^2$) since a quadratic $p(t) \geq 0\; \forall t$ intersects the "real 0" y-axis once or never, so it has one or less real roots

Triangle Inequality $x, y \in \mathbb{R}^n \implies ||x + y|| \leq ||x|| + ||y||$
Proof: $||x + y||^2 = ||x||^2 x \cdot y + ||y||^2 \leq ||x||^2 + 2||x|| \cdot ||y|| + ||y||^2 = (||x|| + ||y||)^2$

2 Basic Topology

Let $f : A \to B$. If for each $y \in B, f^{-1}(y)$ consists of at most one element of A, the f is a *one-to-one* mapping. A is finite if $\exists$ a 1-1 mapping onto a finite set J_n. A is **countable** if $\exists$ a 1-1 mapping onto $\mathbb{N}$.

Theorem Let $\{E_n\}$ be a sequence of countable sets. Then $S = \cup_{n\in\mathbb{N}} E_n$ is countable

Proof: Let all E_n be countably infinite. Consider each E_n on a row, such that set i's jth element is in the i, jth position in an infinite array. However, we can also consider that combining i and j yields a natural number (e.g. 1010 w/ $i = j = 10$) and this integer will be unique for every $i, j \in \mathbb{N}$. Further, each combination will yield an integer greater than 11. Therefore, there some subset of $\mathbb{N}$, call it T, such that there is a 1-1 correspondence from S to T. Because S is a union of countably infinite sets, it must also be countably infinite.

A **metric space** is a set of points X, together with a *metric* $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ s.t i) (pos. definite) $d(x, y) = 0$ iff $x = y$ ii) (symmetry) $d(x, y) = d(y, x)$ iii) triangle inequality w.r.t the metric holds

For a metric space X and $r \in \mathbb{R}^+$, the (open) **ball** $B_r(x)$ is $\{y \in X | d(x, y) < r\}$

$E \subseteq X$ is **open** if $\forall x \in E \; \exists r > 0$ s.t $B_r(x) \subseteq E$

Theorem Open balls are open

Proof: X metrix space, $x \in X$, and $r > 0$. Let $y \in B_r(x)$, $\delta = d(x, y) < r$, and some $\rho \in (0, r - \delta)$.
For any $z \in B_\rho(y)$, $d(x, z) \leq d(x, y) + d(x, z) < \delta + r - \delta = r$. So $B_\rho(y) \subseteq B_r(x)$.

Theorem If $E \subseteq X$ open, $\exists$ family of open balls $\{B_\alpha\}_{\alpha\in I}$ s.t $E - \cup_{\alpha\in I} B_\alpha$

Proof: $\forall x \in E$, $\exists r > 0$ s.t $B_r(x) \subseteq E$. Denote this ball as B_x.
Each $B_x \subset E$, so $\cup_{x\in E} B_x \subset E$. $B_y \ni y \in E$, so $E \subset \cup_{x\in E} B_x$

$E \subseteq X$ is *bounded* if $\exists r > 0, x \in X$ s.t $E \subseteq B_r(x)$

For metric space X and $E \subseteq X$ (*compliment* $E^c = X \backslash E$),
- E°, the **interior** of E, is the set of all $x \in X$ s.t $B_\epsilon(x) \subseteq E$ for some $\epsilon > 0$
- ∂E, the **boundary** of E, is the set of all $x \in X$ s.t $\forall \epsilon > 0$, $B_\epsilon(x) \cap E^c, B_\epsilon \cap E \neq \emptyset$
- E', set of all **limit points** of E, s.t $x \in E'$ iff $\forall r > 0$, $\exists y \in B_r(x)$ where $y \neq x, y \in E$
- $\overline{E}$, the **closure** of E, is the set of all $x \in X$ s.t $\forall r > 0$, $B_r(x) \cap E \neq \emptyset$
- x is **isolated** if $x \in E$ but $x \notin E'$
- E is **dense** in X if $x \in X \implies x \in E'$ and/or $x \in E$

Set Theory Propositions (resulting from above definitions) For any $E \subseteq X$

a) E is open iff $E = E^\circ$
b) $E', \partial E, E \subseteq \overline{E}$
c) $p \in E' \implies B_r(p)$ contains infinitely many points of $E (\forall r > 0)$. E finite $\implies E' = \emptyset$.
d) $\partial E = \overline{E} \backslash E^\circ$
e) $\overline{E} = E \cup E' = E \cup \partial E$
Proof: First note prop b. Then, for $x \in \overline{E}, x \notin E \implies$ every x-ball contains point of E, can't be x itself, so $x \in E'$. Also, every x-ball contains $y \in E^c$, so $x \in \partial E$

$E \subseteq X$ is **closed** if any of the following hold (iff)

a) $E = \overline{E}$
b) $E' \subseteq E$ (Rudin: every limit point is a point of E)
c) $\partial E \subseteq E$
d) E^c is open
Proof: ($a \implies d$) If $x \in E^c$, then $x \notin \overline{E}$, so $\exists r > 0$ s.t $B_r(x) \cap E = \emptyset$ (def of closure) or $B_r(x) \subseteq E^c$
Proof: ($d \implies a$) If $x \in E^c$, $\exists r > 0$ s.t $B_r(x) \subseteq E^c$ or $B_r(x) \cap E = \emptyset$, so $x \notin \overline{E}$ (and $x \notin E$)

Theorem Let $\{G_\alpha\}$ be a collection of open sets. $\cup_\alpha G_\alpha$ is open and finite intersections are open. If $\{G_\alpha\}$ is a collection of closed sets, $\cap_\alpha G_\alpha$ is closed and finite unions are closed

Proof: (open) $x \in \cup_\alpha G_\alpha \implies x \in G_a$ (some a). G_a open $\implies \exists r > 0$ s.t $B_r(x) \subseteq G_a \subseteq \cup_\alpha G_\alpha$. Let $B \subseteq \mathbb{N}$ be finite and consider $\{G_\beta\}_{\beta \in B}$. $x \in \cap_\beta G_\beta \implies x \in G_b \ \forall b \in B$, so $\exists r_b > 0$ s.t $B_{r_b}(x) \subseteq G_b$. Let r be the minimum of all such r. Then $B_r(x) \subseteq \cap_\beta G_\beta$

Theorem: $\overline{E}$ is closed. If $E \subseteq F$ with F closed, then $\overline{E} \subset F$.

Proof: $x \notin \overline{E} \implies \exists B_r(x)$ s.t $B_r(x) \cap E = \emptyset$. Since this is true $\forall x \notin \overline{E}$, $\overline{E}^c$ is open, so $\overline{E}$ is closed. F closed $\implies F' \subset F \implies E' \subset F$ if $E \subset F$. So $E \cup E' = \overline{E} \subset F$
Intersection of all closed sets: from b, $\overline{E} \subset \cap F$. But $\overline{E}$ closed and $E \subset \overline{E}$ so taking $F = \overline{E}$ gives $\cap F \subset \overline{E}$

A, B are **separated** if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. $E \subseteq X$ **disconnected** if $\exists A, B \subseteq X$ nonempty s.t they are separated with $E = A \cup B$. Say A, B *separate* E. A set which isn't disconnected is **connected**

If $A \subseteq X$ closed & open, and $\emptyset \neq A \neq X$, then X disconnected

Proof $\overline{A} = A$, $\overline{A^c} = A^c$ (open $\implies$ compliment closed), and $A \cap A^c = \emptyset$

If A, B separated, E connected, then $E \subseteq A \cup B \implies E \subseteq A$ or $E \subseteq B$

Proof: $E \cap A$ and $E \cap B$ separated because $\overline{E \cap A} \subseteq A$. If both nonempty, E disconnected

If E, F connected and $E \cap F \neq \emptyset$, then $E \cup F$ connected

If E connected can draw paths; path connected $\implies$ connected, not iff (comb and flea)

X metric space, a connected component A of X is a *maximal connected subset*: A connected if $A \subseteq B \subseteq X$ and B connected then $A = B$

Connected components of X partition X: pairwise disjoint, union is X, and are closed. If X has finitely many connected components, they are open.

Sketch: $\forall x \in X$, define A_x as the union of all sets E s.t $x \in X$ and E is connected. Clearly connect all of X, can show A_x connected, and $\forall x, y$, $A_x = A_y$ or $A_x \cap A_y = \emptyset$. Ex: c.c of $\mathbb{Q}$ are singletons $\{q\}(\ \forall q \in \mathbb{Q})$

Every continuous function on $[0, 1]$ is bounded.

An **open cover** of $E \subseteq X$ is a a collection $\{G_\alpha\}_{\alpha \in A}$ of open sets G_α of open sets $G_\alpha \subseteq X$, $E \subseteq \cup_{\alpha \in A} G_\alpha$. A **subcover** is a collection $\{G_\beta\}_{\beta \in B}$ s.t $B \subseteq A$ and it covers E. Cover is finite if A is finite.

$E \subseteq X$ is **compact** if every open cover has a finite subcover.
$\{0, 1, .5, .33, .25...\}$ compact. Take 0 off it's not.

Theorem Compact sets are closed.

Proof: If $x \notin K$, then $\forall y \in K$, let $r_y = d(x, y)/3 > 0$. By triangle inequality (ensured by diving through by 3), $B_{r_y}(x) \cap B_{r_y}(y) = \emptyset$. Infinity problem since boundedness not assumed, so now exploit compactness: $B_{r_z}(y)\}_{y \in K}$ is open cover; let $\{y_i\}_{n=i}^N$ centers of a finite subcover. $V = \cap_{i \in [1,N]} B_{r_{y_i}}(x) \subset K^c$, so K^c open

Theorem Any compact set is bounded.

Proof: Choose any $x \in X$. Then $\{B_n(x)\}_{n \in \mathbb{N}^+}$ is open cover of K. Finite subset of $\mathbb{N}^+$ will suffice, let N be the largest. $K \subseteq B_N(x)$

Theorem Let $K \subseteq X$ compact, $E \subseteq K$ infinite. Then E has limit point in K

Proof: Suppose not. Then every $x \in K$ has B_x s.t $B_x \cap E$ is at most one point (x itself if $x \in K$). $\{B_x\}_{x \in K}$, open cover of K, finite subcover contain at most finitely many (contradiction to infinite assumption).

$E \subset Y$ is **open relative** to Y if for each $p \in E$ $\exists r > 0$ s.t $q \in E \implies d(p,q) < r$. A set is **closed relative** to Y iff its the compliment of a set open relative to Y.

Define an *equivalence set* $[B_r(x_0)]_A$ by $\{x \in A | D(x, x_0) < r\}$

Theorem $E \subset Y \subset X$ is open relative to Y iff (for some $G \subset X$ open) $E = Y \cap G$. E closed rel to y, then E is closed rel to X iff $\overline{E} \subseteq Y$. $E \subseteq Y$ closed rel to $Y \subseteq X$ iff $\exists F \subset X$ closed s.t $E = F \cap Y$.

Proof: (2nd result) $\Longleftarrow$ Know $E = F \cap Y$ for some $F \subseteq X$ closed. Then $\overline{E} \subseteq F$ because F closed and $\overline{E} \subseteq Y$. $E \subset \overline{E} \subset F \cap Y$, so $E = \overline{E}$

Ex: $\sqrt{2} \in \overline{\mathbb{Q}} \subseteq \mathbb{R}$, so $\{x \in \mathbb{Q} | x \in [1,2]\}$, closed in $\mathbb{Q}$, not in $\mathbb{R}$

Theorem If $K \subseteq Y \subseteq X$, then K compact relative to Y iff K compact relative to X

Proof Let $\{V_\alpha\}$ be an open cover relative to Y, and $U_\alpha = V_\alpha \cap Y$. Then $\{U_\alpha\}_{\alpha \in A}$ open cover relative to Y has finite subcover for $B \subseteq A$ finite. $K \subseteq \cup_B U_\beta \subseteq \cup_B V_\beta$ (finite subcover)

Theorem $E \subseteq K$ closed, K compact, then E is compact.

Proof: Let $\{G_\alpha\}$ be an open cover of E. Then union of the open cover and E^c an open cover for K, and $\exists B \subseteq A$ finite that creates a finite cover for both K and therefore E

Finite intersection property (FIP) $\{E_\alpha\}_{\alpha \in A}$ s.t $\cup E_\alpha \subset X$ has FIP if $\forall B \subseteq A$ finite, $\cap_B E_\alpha \neq \emptyset$

Theorem K compact iff $\forall$ families of $\{E_\alpha\}_{a \in A}$ of closed sets with FIP $\cap E_a$ nonempty

Proof: (sketch) same as definition as compact, with *not* everywhere (contrapositive). Take $G_\alpha = K \cap E_\alpha$. Then $\{G_\alpha\}$ open cover iff $K \subseteq \cup_A G_\alpha$ iff $\cap E_\alpha \subseteq \emptyset$ *(FIP means no finite subcover - always be something outside)*

If $\{K_\alpha\}$ family of compact subsets of X with FIP, then $\cap K_\alpha \neq \emptyset$. (Idea: if X compact done. If $\cup K_\alpha$ compact, done. Since care about $\cap K_\alpha$ "all action" happens in each/all K_α)

Proof: Fix any $K_0 \in \{K_\alpha\}_{\alpha \in A}$. Set $E_\alpha = K_\alpha \cap K_0$. $\{E_\alpha\}$ closed subsets of K_0 compact. For $B \subseteq A$, $\cap_B E_\alpha = K_0 \cap (\cap_B K_\alpha)$. Nonempty for B finiifte therefore non-empty for $B = A$.

Theorem Define real numbers $a < b$ then $[a,b]$ compact.

Proof: Suppose $\{G_\alpha\}$ open cover of $[a,b]$ with no finite subcover. So at least $[a,c]$ or $[c,b]$ must have no finite subcover. Define iteratively $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$, so $|b_n - a_n| \leq 2^{-n}|b-a|$. Since $a_n < b$, $x = \sup\{a_n | n \in \mathbb{N}\}$ exists. Notice $a_n \leq x \leq b_n$. Also, any $b_n \in B_{2^{1-n}|b-a|}(x)$. Some G_α contains x, so there exists $r > 0$, $B_r(x) \subseteq G_\alpha$. Take n s.t $2^{1-n}|b-a| < r$ and $[a_n, b_n] \subseteq G_\alpha$.

Any bounded closed subset of $\mathbb{R}$ is compact.

Heine-Borel any closed and bounded subset of $\mathbb{R}^k$ is compact.

(weirstrauss corallary). Any bounded infinite $E \subseteq \mathbb{R}^n$ has limit point.

Theorem If $E \subseteq \mathbb{R}^n$ and every infinite subset of E has limit point, then E compact

Proof: For each $k \in \mathbb{N}$, if $E \cap \{x | x \in (k, k+1)\}$ is nonempty, choose element, call it x_k. Then the collection of these has no limit point (all isolated since we're choosing for each natural number), must be finite (by the Archemedian property since $E \subseteq \mathbb{R}^n$), so E bounded. So let $x \in E'$. For $k \in \mathbb{N}$, choose $x_k \in B_{\frac{1}{k}}(x) \cap E$. Then $\{x_k\}$ infinite for limit $y \in E$. Assume $x \neq y$, let $0 < \epsilon = d(x,y)/3$.

$$d(y, x_k) \geq d(x,y) - d(x, x_k) = 3\epsilon - d(x, x_k) > 3\epsilon - 1/k$$

If $k > 1/\epsilon$, $x_k \notin B_\epsilon(y)$. So $B_\epsilon(y) \cap \{x_k\}$ finite

3 Sequence and Series

Sequence (p_n) is a map $f : \mathbb{N} \to X$, denoted by $f(n) = p_n$. *Range* of sequence is $\{p_n | n \in \mathbb{N}\}$

p_n **converges** to p ($p_n \to p$) if $\forall \epsilon > 0\ \exists N$ s.t $n > N \implies d(p_n, p) < \epsilon$
Difference between limit point and range: $(.5, -3, 1/3, -3, .25, \dots)$ 0 is limit point of range, but not (p_n)

Formally, we say a sequence is *divergent* if the sequence is unbounded or if there is no p that satisfies the above definition (i.e. we do not consider $\pm\infty$ a possible limit point)

$x_n \to x$ iff every ball of x contains all but finitely many x_n

Theorem $E \subseteq X$ is closed iff $\forall$ sequences (x_n) in E $x_n \to x \implies x \in E$
Proof: $x_n \to x$ and $x \in E \implies \exists x_m \in B_r(x)$ (some $m \in \mathbb{N}$ and $\forall r > 0$), so $x \in \overline{E} \implies \forall n \in \mathbb{N}$ choose $x_i \in B_{1/n}(x_n)$, $x_n \to x$ by archemdian property (ball shrinks to "0 radius"). If closed E contains all its lps

Bc of $> N$ definition, infinitely many points clustered around limit. $x \in E$ isolated $\implies x \notin E'$.

Theorem Limit of sequence is unique
Proof: $x_n \to p$ and $x_n \to p \implies \forall \epsilon > 0\ \exists M, N \in \mathbb{N}$ s.t $d(x_n, p) < .5\epsilon$ and $d(x_n, q) < .5\epsilon$. Then by triangle inequality $\forall n > \max\{M, N\}$ $d(p, q) \leq d(p, x_n) + d(q, x_n) = \epsilon$. (use positive definite of metric).

A sequence is *bounded* if its range is bounded.

Theorem Any convergent sequence is bounded
Proof: $x_n \to x \implies \exists N$ s.t $d(x_n, x) < 1 (\forall n > N)$. Then the sequence of distances between x and x_i $i \in [1, N]$ is finite. Let R be the max (well-defined since convergence in $\mathbb{R}$ implicit)

If a_n, b_n sequences and $a_n \geq b_n$ for all but finitely many $n \in \mathbb{N}$, or if $(b_n) = a_N, a_{N+1}, \dots$, then either a_n, b_n both divergent or their limits are equal

$x_n \to x$. If $n > N \implies d(x_n, x) < \epsilon$, then $m > N \implies d(x_n, x_m) < 2\epsilon$

A sequence is **cauchy** if $\forall \epsilon > 0\ \exists N$ s.t $m, n > N \implies (d_n, d_m) < \epsilon$ (convergent $\implies$ cauchy)

A space (X, d) is *(Cauchy) complete* if every cauchy sequence in X is convergent in X

Theorem Compact sets are complete
Proof: Let (x_n) be cauchy in K compact. If $\exists y$ s.t $x_n \leq y$ for infinitely many $n \in \mathbb{N}$, then $x_n \to y$. Assume WLOG no x_n repeat infinitely often. Let $E_N = \{x_n | n > N\}$. Each E_N infinite and $\overline{E}_N \subseteq K$ is compact, has FIP, so $\exists x \in \cap_N \overline{E_N}$. Fix $\epsilon > 0$. Let M s.t $n, m > M \implies d(x_n, x_m) < .5\epsilon$. Then $x \in \overline{E}_M$, so $B_{.5\epsilon}(x)$ contains an element of the sequence. Then the distance between this point and x will be $< \epsilon$

If $X \subseteq Y$ metric space, Y complete and $\overline{X} = Y$, then call (Y, d) the *cauchy completion* of (X, d).

Theorem Every space has a cauchy completion
Proof: (sketch) Points of Y will be equivalence classes of Cauchy sequences (i.e. if we combine sequences, resulting sequence is Cauchy) in X. Define the LIM X_n the formal limit of every Cauchy (X_n) in X. If $x_n \to x$, LIM X_n represents x. Define a metric

$$\overline{d}(LIMa_n, LIMb_n) = \lim d(a_n, b_n)$$

The limit is cauchy in $\mathbb{R}$, so the $\overline{d}$ term exists. Set $LIMa_n = LIMb_n$ if $\overline{d} = 0$. Set $Y = \{LIMx_n | (x_n) \text{ Cauchy sequence} in X\}$. For $x \in X$, identify $x \sim LIM(x)$, so $X \subseteq Y$. Must show $\overline{d}$ is a metric, $d(x, y) = \overline{d}(LIM(x), LIM(y)$, X dense in Y, $(Y, \overline{d})$ is cauchy complete.

x_n sequence in (X,d) and n_k sequence in $\mathbb{N}$ s.t $n_0 < n_1 < \dots$. Then $(x_{n,k})_{k\in\mathbb{N}}$ is a subsequence of (x_n). If $x_{n,k} \overset{k}{\to} y$, call y **subsequential** limit of x_n.
Alternative notation: $A \subseteq \mathbb{N}$, A infinite, $(x_n)_{n\in A}$ is a **subsequence** corresponding to $A = \{n_k | k \in \mathbb{N}\}$

$x_n \to x$ iff every subsequence converges to x. Idea: if $x_n \to x$, points in the subsequence have to be in there at some point (infinite subset of $\mathbb{N}$). If $x \not\to y$ $\exists\,\epsilon > 0$ s.t $\{n \in \mathbb{N} | x_n \notin B_\epsilon(y)\}$ is infinite.

Theorem If x limit point of range (x_n) then x is a subsequential limit of (x_n) (Converse false)
Idea: $x_{n_k} \subset B_{1/k}(k)$. Make some $n_k > n_{k-1}$. This also implies that if (x_n) sequence in compact space K, (x_n) has a convergent subsequence.

Bolzano Weirstruass Bounded sequences in $\mathbb{R}^n$ have convergent subsequences.

Theorem For any seq(x_n), set of all subsequential limits is closed. (but not because E' is closed). Not all subsequential limits are limit points.

Proof: Let $\{A_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ be infinite indexing sets (used to create subsequences) and y_i be a corresponding subsequential limit ($x_n \overset{A_i}{\to} y_i$) with $y_i \to y$. Let A_i^j be the $j-1$th element of the $i-1$th subsequence indexing set and $h_0 = A_0^0$. Define a subsequence by, given n_{k-1}, taking $n_k \in A_k$ $(n_k > n_{k-1})$ s.t $d(x_{n_k}, y_k) < 2^k$.

$$d(x_{n_k}, y) \le d(x_{n_k}, y_k) + d(y_k, y) \le 2^{-k} + \epsilon/2 < \epsilon$$

if you fix $\epsilon > 0$ and take $N \in \mathbb{N}$ s.t $2^{-N} < .5\epsilon$, with $d(y_k, y) < .5\epsilon$ by the definition of a limit.

Sequences in $\mathbb{R}^d$. $\overrightarrow{x}_n \to \overrightarrow{x}$ iff $\lim x_n^i = x^i$ for $1 \le i \le d$. If $\overrightarrow{x}_n \to \overrightarrow{x}$ (likewise for y), then $(\overrightarrow{x} + \overrightarrow{y}) = \overrightarrow{x} + \overrightarrow{y}$. Same for inner product, scalar multiplication.

Monotone sequences. $a_n \le a_{n+1}$ monotone increasing, and monotone decreasing for $\ge$. Strictly monotone if the inequality is not strict.

Theorem Monotone sequence has a limit iff its bounded

The extended real numbers are not a metric space. If $x_n \to \pm\infty$ say it diverges.

A sequence in $\mathbb{R}$ is *unbounded* iff it has a subsequence tending to $\pm\infty$

For $E_N = \{x_n | n \ge N\}$, $(\sup E_n)_n$ is decreasing and $(\inf E_n)_n$ is increasing.
Define $\limsup x_n = \overline{\lim}\ x_n = \lim(\sup E_n)$ and $\liminf x_n = \underline{\lim}\ x_n = \lim(\inf E_n)$

$\limsup x_n$, if finite, is least number α s.t $\forall\epsilon > 0\ \exists\, N$ s.t $n > N \implies x_n < \alpha + \epsilon$

Theorem Let y be the subsequential limit of x_n. Then $y \in [\liminf x_n, \limsup x_n]$

Proof: $y - \limsup x_n = \epsilon > 0$. Then $\exists N$ s.t $\sup\{x_n | n > N\} < \limsup x_n + .5\epsilon$. Thus $n > N \implies x_n < y - .5\epsilon$ and $B_{.5\epsilon}(y)$ contains at most finitely many $(\le N)$ x_n (contradiction). Analogous for $\liminf$.

$\liminf x_n \le \limsup x_n$. (If bounded, B-W y exists. For unbounded, do cases)

For (x_n) in $\mathbb{R}$, $\exists A, B \subseteq \mathbb{N}$ s.t $x_n \overset{A}{\to} \limsup x_n$ and $x_n \overset{B}{\to} \liminf x_n$

Proof: Fix $\epsilon > 0$. Let $\alpha = \limsup x_n$ finite. Take $n_0 = 0$ and given n_{k-1} take $N > n_k$ s.t $\sup_{n\ge N} x_n \in B_\epsilon(\alpha)$. Then take $n_k > N$ s.t $x_{n_k} \ge \sup_{n\ge N} x_n - \epsilon$. If $a = \infty$, range(x_n) not bounded above, has subsequence that limits to ∞, similar for bounded above.

If (x_n) in $\mathbb{R}$, E is a set of subsequential limits in $\mathbb{R}^k$, then $\limsup x_n = \sup E$ and $\liminf x_n = \inf E$.

Proof: let $\alpha = \limsup x_n$. By previous $\alpha \in E$, so $\alpha \leq \sup E$. WTS $\alpha \geq \sup E$. $\forall N \in \mathbb{N}$, let $y_N = \sup_{n \geq \mathbb{N}} x_n \in \mathbb{R}^K$. $\forall n \in \mathbb{N}$, $y_n \geq x_n$. Also $y_n \to \alpha$ by definition, all subsequential limits are cinvergent to α as well. If $x_n \vec{A} x \in \mathbb{R}^K$ then $x \leq \alpha$

3.1 Series

Given (x_n) in $\mathbb{R}^d$ and $N \in \mathbb{N}$, denote $S_N = \sum_{n=0}^{N} x_n$ partial sums. Define infinite sum by $\lim S_N$.

Theorem If the infinite sum conv (*converges* - limit of partial sums exists), then $x_n \to 0$

Converse of above theorem false – the harmonic series $\sum 1/n$ diverges. One way to see this is that the partial sums aren't Cauchy: $|S_{2N} - S_N| = |\sum^{@N} 1/n - \sum^{N} 1/n| = \sum_{N+1}^{2N} 1/n \geq .5N \sum_{N+1}^{2N} 1 = .5$

If $\sum a_n = A$ and $\sum b_n = B, \sum a_n + b_n = A + B$.

Theorem For $r \in \mathbb{C}$,$|r| < 1$ yields $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ (diverges otherwise)

Theorem If $p > 1$, $\sum 1/n^p$ converges; diverges otherwise

Comparison test: let $(a_n), (b_n) \in \mathbb{R}^d$ with $0 \leq ||a_n|| \leq b_n$. $\sum b_n$ convergent $\implies \sum a_n$ convergent.
Proof: set $A_N = \sum^N a_n$. Fix $\epsilon > 0$. Since $\sum b_n$ cauchy, $\exists N$ s.t $\sum_{n=p}^{q} b_n < \epsilon \ \forall N \leq p \leq q$. So $\forall M > N$

$$||A_M - A_N|| = ||a_M + \dots a_{N+1}|| \leq \sum_{i=N-1}^{M} ||a_i|| \leq \sum_{n=p}^{q} b_n < \epsilon$$

Theorem If $(a_n), (b_n) \in \mathbb{R}$ s.t $0 \leq a_n \leq b_n$ and $\sum a_n$ divergent, then $\sum b_n$ divergent.

$n^2 \leq 2^n$. Every geometric series is bounded by $\sum 1/N$ for N sufficiently large.

(Limit comparison test) Let $(a_n), (b_n) \in \mathbb{R}^+$. If the limsup of $a_n/b_n < \infty$ and b_n converges, then $\sum a_n$ converges. If the lim inf of the ratio is positive and $\sum b_n$ diverges, so does $\sum a_n$.
Proof: $\limsup \frac{a_n}{b_n} = R$. Then $\exists N$ s.t $n \geq N \implies \frac{a_n}{b_n} \leq R + 1$. So $a_n < (R+1)b_n$; use comparison test.

Intuition from limit comp test only goes so far: given $\sum x_n$ divergent $\exists (a_n)$ s.t $\limsup \frac{a_n}{x_n} = 0$ and $\sum a_n$ diverges. Given $\sum y_n$ convergent, $\exists (b_n)$ s.t $\liminf \frac{b_n}{y_n} = \infty$ and $\sum b_n$ converges. See HW6 #4 (partial sums).

Big/little o notation: $a_n = O(b_n)$ if $\limsup a_n/b_n < \infty$; $a_n = o(b_n)$ if $\limsup a_n/b_n = 0$

(Ratio Test) If $\limsup a_{n+1}/a_n < 1$ then $\sum a_n$ convergent. If > 1, divergent
Proof: Let $\limsup a_{n+1}/a_n = \lambda < 1$. Take $r = .5(1+\lambda)$. Then $\exists N$ s.t $\forall n > N$ $a_{n+1}/a_n < r$, so $a_{N+M} \leq r^M a_N$ and $\sum_{n=N}^{\infty} a_n \leq \sum_{n+N}^{\infty} a_n r^n$ convergent.

(Root Test) Given $\sum a_n$, set $\lambda = \limsup (a_n)^{1/n}$. Then $\sum a_n$ convergent if $\lambda < 1$, divergent if > 1.
Proof: If $\lambda < 1$, set $r = .5(1+\lambda)$. $\exists N$ s.t $\forall n \geq N$ $a_n \leq r^n$, convergent

Note =1 ambiguous for both, but the root test strictly better. Suppose $\lambda < \infty$ is the ratio test value. Then $\exists N$ s.t $a_{n+m} \leq (\lambda + \varepsilon)^m a_N$. So the root test value is less than $\lambda + \varepsilon$ and thus less than ratio test value.

$\sum a_n$ *converges absolutely* if $\sum ||a_n||$ conv. Convergent $\sum a_n$ *converges conditionally* if $\sum ||a_n||$ doesn't conv

Abs conv stronger than conv and our tests don't like conditional conv (cc always yields root test = 1)

(Alt Series Test) Let (a_n) be a decreasing, non-negative sequence with $\lim a_n = 0$. $\sum (-1)^{n+1} a_n$ conv

Proof: $S_{2N} = (a_1 - a_2) + (a_3 - a_4) + \dots$, $S_{2N+1} = a_1 - (a_2 - a_3) - (a_4 - a_5)$. $0 \le S_{2N} \le S_{2N+1} \le a_1$. Bounded so both conv by monotone convergence. Limit of their difference is the limit of a_{2N+1}, which is 0.

Say $\sum x_n$ conditionally convergent. The sum of only its positive terms is unbounded (same with negative). Idea: divide the terms between $a_n > 0$ and $b_n < 0$. Thus $\sum |x_n| = \sum a_n - \sum b_n$ diverges. $\sum a_n$ and $\sum b_n$ can't both converge, in fact both diverge.

Define $k(n) : \mathbb{N} \to \mathbb{N}$ 1-1. Then for $b_n = a_{k(n)}$, $\sum b_n$ is a *rearrangement* of $\sum a_n$

Theorem If $\sum a_n = A$ converges absolutely, then every rearrangement of $\sum a_n$ also converges to A.
Proof: Let r be the rearrangement bijection and fix $\epsilon > 0$. Then $\exists N$ s.t $\sum_{n=N}^{\infty} |a_n| \le \epsilon$. Take $M = \max_{i \in [1,N]} r(N)$. Then $|\sum_{i=M}^{\infty} a_{r(n)}| \le \sum_{n=N}^{\infty} |a_n| < \epsilon$.

Theorem (Riemann rearrangement) If $\sum x_n$ is conditionally convergent, there exists rearrangements where the rearranged series converges to $\pm\infty$ and any real number.
Idea: Let A and B be the set of indices of pos/negative terms respectively. Define $a_n = x_m$, where m in the nth term in A, and $b_n = -x_m$ similarly. (e.g. $A = \{3, 7, 11..\} \to (x_3, x_7, x_{11})$). Know $x_n \to 0$, so same for a_n, b_n. Take N s.t $n > N \implies b_n < .5$. Take t_k s.t $\sum_{n=1}^{t_k} a_n > k$. (bc sum diverges). For $n > t_k + N + k$, partial sum is greater than $C - k + k/2 \to \infty$ (sum of b_1 to b_N is C). For $\lambda \in \mathbb{R}^+$, take # of a_n to get beyond λ, then minimum number of b_n to go back, and so on. Similar consideration for $\lambda \in R^-$

Let $(a_n) \in \mathbb{C}$. A **power series** is a function which assigns to $z \in \mathbb{C}$ the series $\sum a_n z^z = a_0 + a_1 z + \dots$. For $D = \{z | \sum a_n z^n \text{ convergent}\}$, the power series defines a function $D \to \mathbb{C}$.
Notable Examples: $\sum z^n/n! = e^z$, $(-1)^n z^{2n+1}/(2n+1)! = \sin(z)$

Theorem For $\sum a_n z^n$ power series, $\exists R \in [0, \infty]$ s.t $\sum a_n z^n$ converges absolutely in $|z| < \mathbb{R}$ and diverges otherwise. Define *radius of convergence* as $R = \limsup |a_n/a_{n+1}|$ (if converges, otherwise $R = (\limsup |a_n|^{1/n})$

Convergence on boundary nebulous; if the components converge then the whole thing does

For $p \in [1, \infty]$, the **p-norm** on $\mathbb{R}^n$ is $||\vec{x}||_p = (|\sum x_i|^p)^{\frac{1}{p}}$.

Theorem For any $x \in \mathbb{R}^d$ and $1 \le p \le q \le \infty$ $||x||_q \le ||x||_p \le d^{\frac{1}{p} - \frac{1}{q}} ||x||_q$.

Define the metric space (X, d) to be with respect to the p-norm. Most properties (open, convergent, compact, cauchy, connected) hold in the 2-norm iff they hold in the p-norm

Infinite dimensional vector spaces: Let $\cup_{n=1}^{\infty} \mathbb{R}^n = \mathbb{R}^{\mathbb{N}}$
Idea: $\mathbb{R} \subseteq \mathbb{R}^2$. So generalizing this further, $n < m \implies \mathbb{R}^n \subseteq \mathbb{R}^m$ by $(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$.

Let $p \in [0, \infty]$ with $q \in \mathbb{R}$. In $(\mathbb{R}^{\mathbb{N}}, p)$, $(\frac{1}{n^q})$ is Cauchy if $pq > 1$ and unbounded if $pq \le 1$. Bounded, not Cauchy if $pq = \infty \cdot 0$.
Proof: $||a_n||_p = \left[\sum_{i=1}^{n} \frac{1}{i^n}\right]^{1/p}$ and for $n < m$ $||a_m - a_n||_p = \left[\sum_{i=n+1}^{m} \frac{1}{i^n}\right]^{1/p}$. For $p = \infty$, $||a_n||_\infty$ is 1 if $q \ge 0$ and n^{-q} otherwise. So bounded iff $q \ge 0$. $||a_n - a_m||_\infty = \max\{n^{-q}, m^{-q}\}$, so Cauchy iff $1/p \to 0$

Theorem For $p \in [1, \infty]$ $(\mathbb{R}^{\mathbb{N}}, p)$ is not complete.

$\mathbb{R}^\infty$ is the set of all sequences in $\mathbb{R}$. Identify $(x^1, \dots, x^d) \in \mathbb{R}^{\mathbb{N}}$. Now we can formally consider $\mathbb{R}^{\mathbb{N}}$ sequences with finitely many non-zero terms. Moreover, $\mathbb{R}^\infty$ is a vector space

For $p \in [1, \infty], a \in \mathbb{R}^\infty$, denote $||a||_p = |\sum_{n=0}^{\infty} (a^n)^p|^{1/p} \in \mathbb{R}^*$ and $||a||_\infty = \sup_n |a^n| \in \mathbb{R}^*$.
For $p \in [1, \infty]$, denote $\ell^p - \{a \in \mathbb{R}^\infty \big| \, ||a||_p \le \infty\}$. Then ℓ^p is a metric space with p-norm.
And for any $p \in [1, \infty], \mathbb{R}^{\mathbb{N}} \subseteq \ell^p \subseteq \mathbb{R}^\infty$

Let $a \in \mathbb{R}^\infty$, $a_n = (a^0, \dots, a^n, 0, 0, \dots) \in \mathbb{R}^\mathbb{N}$. Then $a \in \ell^p$ $(p < \infty)$ iff $a_n \to a$ in ℓ^p

Sketch: $a \in \ell^p$ iff $\sum |a^n|^p$ conv. $a_n \to a$ iff $\sum_N^\infty |a^n|^p \xrightarrow{N} 0$

Theorem For $p \in [1, \infty)$, ℓ^p is the completion of $\mathbb{R}^\mathbb{N}$ with the ℓ^p norm.

Proof: Let $(a_n) \in \ell^1$ be Cauchy. WTS $\exists a \in \ell^1$ s.t $a_n \to a$. Let a_n^k be the kth element of the nth sequence. Let $a_n^k \xrightarrow{n} a^k \in \mathbb{R}$ (can do this since because partial sums will be cauchy). Now define $a = (a^1, a^2, \dots) \in \mathbb{R}^\infty$. We know it's in ℓ because we can define R strictly greater than its norm (so will be greater than sum of first k components for all k). Cannot take straightforward limit because each k needs different N. However, we can define a N with respect to $||a_n - a_m||$, use an absolute sum taking a limit in m, then taking an infinite sum with sup over all n, giving us what we want.

4 Continuity

Let $f : X \to Y$ a function. we denote Y the codomain for $E \subseteq X$

$f(x)$ is the **image**. For $E \subset Y$, the **inverse** image $f^{-1}(E) = \{x \in X | f(x) \in E\}$

For $f : E \to y$ with $p \in E'$ and $E \subset X$, say $\lim\limits_{x \to p} f(x) = y$ if $\forall \varepsilon > 0 \ \exists\, \delta > 0$ s.t $x \in E$ with $x \neq p$,

$$d_X(x, p) < \delta \implies d_Y(f(x), y) < \varepsilon$$

Let $S^1 = \{z \in \mathbb{C} | |z| = 1\}$ (unit circle) and define $\arg : S^1 \to [0, 2\pi)$. $\lim_{z \to 1} \arg$ DNE.
$\forall \delta > 0$, $e^{i\frac{\delta}{2}}, e^{i(2\pi - \frac{\delta}{2})} \in B_\delta(1)$. But $d(x, y) = 2\pi > \varepsilon$

$p \in E'$, $\lim\limits_{x \to p} f(x) = y$ iff $\forall (x_n) \in E$, $\lim_n x_n = p$, $x_n \neq p \ \forall n \implies \lim f(x_n) = y$

Proof: ($\impliedby$) Suppose the limit is not y: then $\exists \varepsilon > 0$ s.t $\forall \delta \ \exists\, x \in B_\delta(p)$ where $d(f(x), y) \geq \varepsilon$. For each n, take $x_n \in B_\delta(p)$ s.t $d(f(x_n), y) \geq \varepsilon$. Then $f(x_n)$ does not converge to y

So limits are unique: a and b both $\lim\limits_{x \to p} f(x) = y$, $a = b$. "Vacuously" continuous at isolated points

f is **continuous** at $p \in E$ if $\forall \varepsilon > 0 \ \exists\, \delta > 0$ s.t $\forall x \in E$ where $d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \varepsilon$

Earlier limit definition: undefined if $p \notin E'$. Continuity guarantees everything in a small enough nbhd is a lp

Theorem $f : X \to Y$ continuous iff $\forall U \subset X$ open $\implies f^{-1}(U) \subset X$ open

f continuous iff pre-images of closed sets are closed

Theorem Composition of continuous functions are continuous
WTS $X \xrightarrow{f} Y \xrightarrow{g} Z$

Proof: (1): Fix $\varepsilon > 0, p \in X$. Since g continuous, $\exists \eta > 0$ s.t $d_Y(f(p), y)) < \eta \implies d_Z(g \circ f(p), g(y)) < \varepsilon$. Since f continuous, $\exists \delta > 0$ s.t $d_X(p, x) < \delta \implies d_Y(f(p), f(x)) < \eta \implies d_Z(\circ, \circ) < \varepsilon$

Proof: (2) Let $x_n \to p \in X$. Then $f(x_n) \to f(p)$. $g(f(x_n)) \to g(f(p))$.

Proof: (3) Let $U \subset Z$ open. Then $g^{-1}(U) \subset Y$ open, so $f^{-1}(g^{-1}(U)) \subset X$ open

If $\lim\limits_{x \to p} f(x) = y$ for $f : E \to Y$, $p \in E'$, and $g : Y \to Z$ cont $\lim\limits_{x \to p} g \circ f(x) = g(y)$

Theorem $+ : \mathbb{R}^2 \to R$ s.t $+(a, b) = a + b$ is continuous

Proof: $+((x_n, y_n)_n) \to x_n + y_n \to x + y = +((x, y))$

A function is bounded if its image is bounded. Continuous function on a compact domain is bounded.

Theorem For $f : K \to Y$ continuous, K compact, $f(K)$ is compact

Proof: Let $\{G_a\}_a$ open cover of $f(K)$. WTS $\{f^{-1}(G_a)\}_a$ open cover of K. Each $f^{-1}(G_a)$ is open. For $x \in K$, $f(X) \in f(K)$ so $\exists \alpha$ s.t $f(x) \in G_\alpha$, so $x \in f^{-1}(G_\alpha)$.
So $\exists A$ finite s.t $K \subset \cup_{a \in A} f^{-1}(G_a)$. $\forall y \in f(K) \ \exists\, x \in K$ s.t $y = f(x)$. So $\exists alpha \in A$ s.t $x \in f^{-1}(G_\alpha)$, so $y = f(x) \in G_\alpha$. So $\{G_\alpha\}_{\alpha \in A}$ is a finite subcover.

Let $f : K \to Y$ surjective, K compact. If $U \subset K$ open, then $f(U)$ open. If f bijection, $f^{-1} : Y \to K$ continuous

A function $f : X \to Y$ is **uniformly continuous** if $\forall \varepsilon > 0 \exists \delta > 0$ s.t $d_x(x, y) \implies d_y(f(x), f(y))$

Theorem Let $f : X \to Y$ uniformly continuous, (x_n) cauchy in X. $(f(x_n))$ Cauchy in Y

Proof: Fix $\varepsilon > 0$. $\exists \delta$ s.t $d_x(a, b) < \delta \implies d_Y(f(a), f(b)) < \varepsilon$. Moreover, $\exists N$ s.t $m, n > N \implies d_X(f(x_n), f(x_m)) < \delta \implies d_Y(f(x_n), f(x_m))$

Theorem $f : K \to Y$ continuous, K compact, then f uniformly continuous.

Proof: Fix $\varepsilon > 0$. $\forall x \in K$, define r_x s.t $B_{r_x}(x) \subset f^{-1}(B_{\frac{\varepsilon}{2}} f(x))$, so $\{B_{\frac{r_x}{2}}(x)\}$ open cover. Over finite subcover, take $\delta = \min \frac{r_{x_n}}{2}$. If $d_X(x,y) < \delta$ $\exists x_n$ s.t $d(x, x_n) < \frac{r_{x_n}}{2}$. So $d(x_n, y) < r_{x_n}$ and $d_Y(f(x), f(y)) < \varepsilon$

Theorem Let $E \subseteq X$, $f : E \to Y$, $p \in E$. If f unif cont and Y complete, then $\lim_{x \to p} f(x)$ exists.

Proof: Let $(a_n) \in E$, $a_n \to p$, $a_n \neq p$ $\forall n$. Then (a_n) cauchy, $(f(a_n))$ Cauchy, limit exists call it y. Let $(b_n) \in E$, $b_n \to p$, $b_n \neq p$ $\forall n$. Then $(a_0, b_0, a_1, b_1, \dots) \to p$, cauchy. By thrm the seq-function converges to y

Theorem Let $E \subseteq X$. $f : E \to Y$ uniformly continuous, Y complete. Then $\exists \overline{f} : \overline{E} \to Y$ continuous where $f(x) = \overline{f}(x)$ $\forall x \in E$ (see HW for proof)

Theorem $f : X \to Y$ continuous, X connected, then $f(X)$ connected

Proof: Proof by contradiction: $A, B \subseteq Y$, $f(x) \subseteq A \cup B$, $\overline{A} \cup B = \overline{B} \cup A = \emptyset$. Then $X \subseteq f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A), f^{-1}(B)$ disjoint.

(IVT) $f : X \to \mathbb{R}$ continuous,X connected , $\exists a, b \in X$ s.t $r \in (f(a), f(b)) \implies \exists c$ s.t $f(c) = r$

Theorem E connected iff $\forall f : E \to \{0, 1\}$ continuous $\implies f(E) = 0$ or $f(E) = 1$

Proof: ($\Longleftarrow$) E disconnected, so $E = A \cup B$, disjoint, clopen, nonempty.

A set E is *path-connected* iff $\forall a, b \in E$ $\exists f : [0, 1]] \to b$ continuous s.t $f(0) = a, f(1) = b$.
Theorem Path-connected iff connected

For $f : \mathbb{R} \to \mathbb{R}$, $p \in \mathbb{R}$, define $g : (-\infty, p) \to \mathbb{R}$ and $h(p, \infty)$ equal to f on their domains. limit "from the right" $x \to p^+$ is defined as $\lim_{x \to p} h(x)$, similarly "from the left" $x \to p^-$ with g. Cont iff limits equal

"Simple" *discontinuity*: removable ("hole") & jump (e.g piece wise). "Essential": limit from left or right DNE

Theorem Monotone functions can't have infinitely many or non-jump discontinuities

5 Derivatives ($f : \mathbb{R} \to \mathbb{R}$)

Derivative of f at $a \in \mathbb{R}$ is $f'(a) = \frac{d}{dx} f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

If f derivative exists $\forall a \in U$ open, f is *differentiable* on U

Theorem If f is differentiable at a, then f is continuous at a
 Proof: Since $f'(a)$ exists, $\lim_{x \to a} f(x) - f(a)$

Theorem (Prod/Quotient Rules) $(fg)'(p) = f'(p) \cdot g(p) + f(g) \cdot g'(p)$ and $(1/f)'(p) = -f'(p)/f(p)^2$

Theorem (Chain Rule) $f : U \to \mathbb{R}, f(U) \subset V$ open,$g : V \to \mathbb{R}$. Assuming existance, $(g \circ f)'(p) = g' \circ f(p) \cdot f'(p)$

Theorem (Rolle's) $f : [a, b] \to \mathbb{R}$ cont and diff. If $f(a) = f(b)$, then $\exists c \in (a, b)$ s.t $f'(c) = 0$

(MVT) Corollary to Rolle's without $f(a) = f(b)$ condition: $\exists c \in (a, b)$ s.t $f'(c)(b - a) = f(b) - f(a)$

$f : X \to Y$ is **Lipschitz**:if $\exists M > 0$ s.t $\forall x, y \in X d_Y(f(x), f(y)) \leq M d_X(x, y)$

Theorem For $f : U \to \mathbb{R}$ diff w/ U cnctd, if $\exists M \geq 0$ s.t $|f'(x)| \leq M \forall x \in U$, f is lipschitz

Lipschitz gives some information on the rate of convergence, more than just differentiable at a point
Theorem If f diff at p, $\exists M \geq 0, r > 0$ s.t $|f(x) - f(p)| \leq M|x - p| \ \forall x \in B_r(p)$
BUT Lipschitz weaker than continuity in terms of $\implies$ differentiability.;$f(p) + f'(p)(x - p)$ is tangent line.

$a + b(x - p)$ is the *best linear approx* of f at p if $\lim_{x \to p} \left| \frac{f(x) - [a + b(x-p)]]}{x - p} \right| = 0$

Theorem For $f : U \to \mathbb{R}$ cont at $p \in U$ best linear approx iff f diff at p, $a = f(p), b = f'(p)$

With $f^{(n)}(x) = \frac{d^n}{dx^n} f(x)$, for f n-diff at P, nth degree **Taylor polynomial** is $T_p^n(f)(x) = \sum_{i=0}^n \frac{1}{i!} f^{(i)}(p)(x-p)$
As n gets larger, the error between the approximation and the function goes to 0

Let $f(b) = f(a) = f'(a) = ... = f^{n-1}(a) = 0$, $f^n(c) = 0$ (consider Rolle's), then $\exists c$ s.t $f^{(n)}(c) = 0$
Theorem (Taylor) Let $\alpha, \beta \in (a, b)$. $\exists z \in (\alpha, \beta)$ s.t $f(\beta) = T_\alpha^{n-1} + \frac{f^{(n)}(z)}{n!}(\beta - \alpha)^n$

Theorem (L'Hopital) For $f(x) = g(x) = 0$, existence of deriv, and $g'(x) = 0$. $\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$
Theorem If $f : U \to \mathbb{R}$ n-times diff at $p \in U$, and $p : \mathbb{R} \to \mathbb{R}$ degree n-polynomial then $\lim_{x \to o} |f(x) - q(x)|/|(x - p)^n| = 0$ iff $Q = T_p^n(f)$
 Proof: By L'hopital (see Rudin for a stronger statement)

$$\lim_{x \to p} \frac{f(x) - T_p^n(f)(x)}{(x - p)^n} = \cdots = \lim \frac{f^{(n-1)}(x) - f^{(n-1)}(p) - f^{(n)}(p)(x - p)}{n!(x - p)} = 0$$

If Q is best polynomial approximation, add and subtract $T_p^n(f)(x)$ from given limit so $\frac{T_p^n - Q}{(x-p)^n} \to 0$. But $[T_p^n(f) - Q](x) = a_0 + a_1(x - p) + \cdots + a_n(x - p)^n$ then limit converges iff all the coefficients are equal.

The **Taylor series** for f at p is the power series $T_p(f)(x) = \sum \frac{f^{(n)} * p)}{n!}(x - p)^n$.

If $\exists R > 0$ s.t $T_p(f)(x) = f(x) \ \forall x \in B_R(p)$ call f *analytic* at p. Are infinitely diff.

Theorem Let $f : [a, b] \to \mathbb{R}^n$ cont diff on (a, b) w/ $||f'|| \leq M$. $||f(b) - f(a)|| = M|b - a|$
 Proof: Let $v \in \mathbb{R}^n$. By MVT $|v(b - a)^{-1}(f(b) - f(a)| \leq \sup(v \cdot f') \leq ||v||M$. Take $v = f(b) - f(a)$

6 Integration

A **partition** of $[a,b] \subseteq \mathbb{R}$ is a finite set of points $\mathcal{P} = \{x_i\}_{i=0}^n$ s.t $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$

Given partitions P, Q if $P \subseteq Q$, Q is a *refinement* of P. $P \cup Q$ is *common refinement*

Given partition P, if $t_i \in [x_{i-1}, x_i]$ $(i \in [1,n])$, P is a *tagged partition* with tags $\{t_i\}$

Given tagged partition P, let $\Delta_i = x_i - x_{i-1}$, $I_i = [x_{i-1}, x_i]$, and $S(f,P) = \sum_{i=1}^n f(t_i)\Delta_i$ be the Riemann sum. Define $m_i = \inf\{f(x) | x \in I_i\}$ and $M_i = \sup\{f(x) | x \in I_i\}$. Define the upper and lower **Darboux sums** by $L(f,P) = \sum_{i=1}^n m_i \Delta_i$ and $U(f,P) = \sum_{i=1}^n M_i \Delta_i$. Note $L(f,P) \leq S(f,P) \leq U(f,P)$.

Denote the *lower Darboux integral* by $\underline{\int_a^b} f = \sup_P L(f,P)$ and similarly for upper integral. If they are both equal to some $C \in R$, f is **integrable** and $\int_a^b f - C$

Given $\alpha : [a,b] \to \mathbb{R}$ increasing and P part, denote $\alpha_i = \alpha(x_i)$ $(x_i \in P)$. Define $L(f,P,\alpha) = \sum_{i=1}^n m_i[\alpha_i - \alpha_{i-1}]$ and $\underline{\int_a^b} f d\alpha = \sup_P L(f,P,\alpha)$ (similar for $U(f,P,\alpha)$. If equality exists, common value $\int f d\alpha$ is **Stieltjes** integral and say $f \in \mathcal{R}(\alpha)$. Notice $\alpha(x) = x$ yields Darboux. Define $f \in \mathcal{R}$ Darboux integrable.

Let $f : [a,b] \to \mathbb{R}$ bounded, P, Q part s.t. $P \subseteq Q$. Then $L(f,P,\alpha) \leq U(f,Q,\alpha)$ and $L(f,Q,\alpha) \leq U(f,P,\alpha)$

The above result yields for *any* partitions P, Q $L(f,P,\alpha) \leq U(f,Q,\alpha)$
Proof: $L(f,P,\alpha) \leq L(f,P \cup Q,\alpha) \leq U(f,P \cup Q,\alpha) \leq U(f,Q,\alpha)$

$f \in \mathbb{R}(\alpha)$ iff $\forall \varepsilon > 0$ $\exists P$ s.t $U(f,P,\alpha) - L(f,P,\alpha) < \varepsilon$

Theorem If $f : [a,b] \to \mathbb{R}$ cont, α inc, then $f \in \mathbb{R}(\alpha)$
Proof: Let $L = (\alpha(b) - \alpha(a))^{-1}$ and fix $\varepsilon > 0$. $\exists \delta > 0$ s.t $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Let P s.t $\Delta_i < \delta$ $\forall i$. Then $M_i - m_i \leq L\varepsilon$, so $[U - L](f,P,\alpha) \leq L\varepsilon(\alpha_i - \alpha_{i-1}) \leq \varepsilon$

Theorem Let $g : [a,b] \to \mathbb{R}$ bounded, $f : \mathbb{R} \to \mathbb{R}$. If $g \in \mathbb{R}(\alpha)$, f cont, then $f \circ g \in \mathcal{R}(\alpha)$
Proof: If P s.t $[U - L](g,P,\alpha)$ small, then $\forall i$ either $M_i - m_i$ small, or $\alpha_i - \alpha_{i-1}$ small. If $M_i - m_i$ small, $f(M_i) - f(m_i)$ small; f doesn't matter much. WLOG f uniformly continuous (since g bounded). Take ε, δ s.t $\sup_{I_i} g(x) - g(y) < \delta \implies \sup f \circ g(x) - f \circ g(y) < \varepsilon$. $[U - L](g,P,\alpha) \geq \sum_{i \text{ s.t } M_i - m_i > \delta} \delta(\alpha_i - \alpha_{i-1})$. Let $K = \sup f \circ g - \inf f \circ g$. Then $[U - L](f \circ g, P, \alpha) \leq \varepsilon[\alpha(x) - \alpha(y)] + \sum_{M_i - m_i > \delta} K[\alpha_i - \alpha_{i-1}] \leq \varepsilon[\alpha(x) - \alpha(y)] + [U - L](g,P,\alpha)K\delta^{-1}$. This bound holds for all partitions so $\exists P$ s.t $[U - L](g,P,\alpha)K\delta^{-1} = 1$

Theorem Stieltjes integration preserves monotonicity (α incr)

Theorem Let $\phi : \mathbb{R} \to \mathbb{R}$ cont and strictly incr. $f, \alpha : [\phi(a), \phi(b)] \to \mathbb{R}$, $f \in \mathcal{R}(\alpha)$. Then $g = f \cdot \phi$ and $\beta = \alpha \cdot \phi$ yield $g \in \mathcal{R}(\beta)$. Moreover, $\int_a^b g d\beta = \int_{\phi(a)}^{\phi(b)} f d\alpha$
Proof: For P partition of $[a,b]$, $\Phi(P) = \{\phi(x_i)\}$ is a partition of $[\phi(a), \phi(b)]$. $U(f, \Phi(P), \alpha) = U(g, P, \beta)$..

Consider α diff on $[a,b]$. $\alpha_i - \alpha_{i-1} = \alpha_i - \alpha_{i-1} \cdot (\Delta_i / (x_i - x_{i-1})) = \alpha'(t_i)\Delta_i$ (some $t_i \in I_i$)

Theorem Let $f, \alpha : [a,b] \to \mathbb{R}$ with $\alpha' : (a,b) \to \mathbb{R}$ cont. $f \in \mathcal{R}(\alpha)$ iff $f\alpha' \in \mathcal{R}$ and $\int f d\alpha = \int f\alpha'$

If $f(x) = g(x)$ $\forall x \neq p$ then $\int f d\alpha = \int g d\alpha$ if α cont at p

Given $f : [a,b] \to \mathbb{R}$, $\mathcal{P}$ partition, the *variation* of f over $\mathcal{P}$ is $V(f, \mathcal{P}) - \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$. The *total variation* of f is $TV(f) = \sup_{\mathcal{P}} V(f, \mathcal{P})$. If $TV(f)$ is finite, f has **bounded variation**; "f is BV"

If f monotone, $TV(f) = |f(b) - f(a)|$. So the sum of montone functions (on a closed interval) is BV

Theorem (Jordan Decomposition) f is BV iff its equal to the sum of inc and dec fun.

Proof: ($\implies$) $f_t = f$ restricted to $[a.t]$. Then $TV(f_x)$ increasing. Given $c < d$, let $\mathcal{P}$ partition of $[c,d]$. $f(d) - f(c) = \sum f(x_i) - f(x_{i-1}) \leq \sum |f(x_i) - f(x_{i-1})| = TV(f_d) - TV(f_c)$. $f(d) - TV(f_d) \leq f(c) - TV(f_c)$. So $f(x) = TV(f_x) + [f(x) - TV(f_x)]$, a sum of incr and decr.

Let $f, \alpha : [a,b] \to \mathbb{R}$, α BV, $\mathcal{P}$ partition s.t. $M_i - m_i < \varepsilon$ with $\{t_i\}, \{s_i\}$ tags. Then $|\sum f(t_i)[\alpha_i - \alpha_{i-1}] - \sum f(s_i)[\alpha_i - \alpha_{i-1}]| \leq \varepsilon TV(\alpha)$. So integrate with BV weights if α monotonic. If α (BV) not monotonic, use JD such as: $\alpha = \alpha_+ - \alpha_-$, incr and decr function. Define $\int d\alpha = \int f d\alpha_+ - \int f d\alpha_-$. So $\mathcal{R}(\alpha) = \mathcal{R}(\alpha_+) \cap \mathcal{R}(\alpha_-)$. Formalizing this, let α BV. If $f \in \mathcal{R}(\alpha)$, $\forall \varepsilon > 0$ $\exists \mathcal{P}$ tagged partition s.t $|\int f d\alpha - \sum(t_i)p[\alpha_i - \alpha_{i-1}]| < \varepsilon$

Proof: Use use above decomposition, triangle ineq, and then $\mathcal{P}$ that works for both α_+, α_-

(FTC 1) Let $f \in \mathcal{R}$ on $[a,b]$ and $F(x) = \int_a^x f$. For, F cont and f cont at p, $F'(p) = f(p)$

Proof: Since f bounded, $|f| \leq M$ (some M, so $|\int_a^x f - \int_a^y f| = |\int_y^x f| \leq M|x-y|$, so Lipschitz continuous. So $F(x) - F(p) = \int_p^x f = \int_p^x f(p) + \int_p^x [f - f(p)] = f(p)(x-p) + E$. So $f(p) = f'(p)$ iff $E = o(x-p)$. If f cont at p, for $\varepsilon > 0$ $\exists \delta > 0$ s.t $x = B_\delta(p) \implies |f(x) - f(p)| < \varepsilon$ and $|E| \leq \varepsilon |x-p|$

(FTC 2) Let $f \in \mathcal{R}$ on $[a,b]$ and $f : [a,b] \to \mathbb{R}, f' \in \mathcal{R}$. Then $f(b) = f(a) + \int_a^b f'$

Proof: By MVT, $\exists$ tags of P s.t $\sum f'(t_i)\Delta_i = \sum f(x_i) - f(x_{i-1}) = f(b) - f(a)$. Thus $L(f', f) \leq f(b) - f(a) \leq U(f', f)$. Assume $L - U \to 0$

(Int by Parts) For $f.g : [a,b] \to \mathbb{R}$ w/ $f', g' : [a,b] \to \mathbb{R}\,(\in \mathcal{R})$, $\int f \cdot g' = fg|_b^a - \int f' \cdot g$

Proof: Product rule + FTC: $\int (f \cdot g)' = \int f \cdot g' + \int f' \cdot g = f(b)g(b) - f(a)g(a)$

To bring Stieltjes in (doesn't apply to FTC in general), consider $f \in \mathcal{R}(\alpha)$, f diff. For P partitions

$$\sum [(f\alpha)_i - (f\alpha)_{i-1}] = \sum f_i[\alpha_i - \alpha_{i-1}] + \sum [f_i - f_{i-1}]\alpha_{i-1} =\| + \sum \alpha_{i-1} f'(t_i)\Delta_i \approx \int f d\alpha + \int \alpha f'$$

Theorem $f : [a,b] \to \mathbb{R}$, diff on (a,b), α BV. If $f \in \mathcal{R}(\alpha)$, $f'\alpha \in \mathcal{R}$, then Int by parts holds ($\int_b^a f d\alpha$)
Stieltjes integral is roughly weighted Riemann int of derivative

7 Sequences of Functions

For X, Y metric spaces, $\mathcal{F}(X,Y)$ denote set of all $X \to Y$.

For (f_n) seq in $\mathcal{F}(X,Y)$, say $f_n \to f$ **pointwise** if $\lim f_n(x) = f(x) \;\; \forall x$

For (f_n) seq in $\mathcal{F}(X,Y)$, say $f_n \to f$ **uniformly** ($f_n \rightrightarrows f$)if $\forall \varepsilon > 0 \;\; \exists N$ s.t $n \geq N \implies d(f_n(x), f(x)) < \varepsilon \; (\; \forall x \in X)$

$f_n \rightrightarrows f$ iff $\sup_{x \in X} d(f_n(x), f(x)) \to 0$

Theorem (Cauchy Criteria): Let Y complete, $(f_n) \in \mathcal{F}(X,Y)$ s.t $\forall \varepsilon > 0, \exists N$ s.t $n, m > N \implies \sup d(f_n, f_m) < \varepsilon$. Then $\exists f \in \mathcal{F}(X,Y)$ s.t $f_n \rightrightarrows f$

Proof: For x fixed, $f_n(x)$ cauchy, lim exists, call it $f(x)$. Fix $\varepsilon > 0$. Choose N s.t $m, n > N \implies \sup_x d(f_n(x), f_m(x)) < \varepsilon$. Then $d(f_n(x), f(x)) = \lim_{m \to \infty} d(f_n(x), f_m(x)) \leq \varepsilon \; (\forall x)$ holds for $n > N_\varepsilon$, independent of x.

Theorem (Weierstrass M-test) For $f_n \in \mathcal{F}(X, \mathbb{R})$, if $\forall n \exists M_n \in \mathbb{R}$ s.t $\sup_x |f_n| \leq M_n$ with $\sum M_n < \infty$. Then $\sum f_n$ converges uniformly.

Theorem Let $E \subseteq X$, $f_n, f : E \to Y$, $p \in E'$. If $f_n \rightrightarrows f$ and $\lim_{x \to p} f_n(x) = L_n \in Y$, then (L_n) cauchy and if $\lim L_n$ exists, $\lim L_n = \lim_{x \to p} f(y)$. If $f_n : X \to Y$ cont and unif conv, then f cont.

Proof:

$$d(L_n, L_m) \leq d(L_n, f_n(x)) + d(L_m, f_m(x)) + \sup d(f_n, f_m)$$

Choose N s.t 3rd term $< \varepsilon$. Choose x s.t $1/2 < \varepsilon$ (dependent on n, m), so cauchy. Define $L = \lim L_n$

$$d(f(x), L) \leq \sup d(f, f_n) + d(f_n(x), L_n) + d(L_n, L)$$

Let X, Y metric spaces w/ Y cplt. Define $C^0(X,Y)$ the set of bounded, cts $X \to Y$ w/ $d_0 = \sup_x d_Y(f(x), g(x))$

C^0 is complete and $\int$ is a continuous function $C^0([a,b]) \to \mathbb{R}$

Let α BV, $f_n \rightrightarrows f$, $f_n \in \mathcal{R}(\alpha)$. $f \in \mathcal{R}(\alpha)$, $\int f_n d\alpha \to \int f d\alpha$

Proof: WLOG α increasing. Let $\varepsilon > 0$, take n s.t $\sup |f_n - f| < \varepsilon$. Take P s.t $U - L < \varepsilon$. $\sup f \leq \sup f_n + \varepsilon$ (over each i, similar for a lower bound with inf).

$$[U - L](f, P, \alpha) = \sum (M_i - m_i)(\alpha - \alpha_{i-1}) \leq (M_i^n - m_i^n + 2\varepsilon)(\alpha - \alpha_{i-1}) \leq \varepsilon + 2(\alpha(b) - \alpha(a))\varepsilon$$

Consider $f_n(x) = f_n(a) + \int_a^x f_n' \to f(a) + \int_a^x f' = f(x)$ ($f_n' \in \mathbb{R}$, $f_n \to f$ pointwise & $f_n' \to g$ uniform, f' cont). So $f'(x) = g(x)$ if g continuous

Let (f_n) diff, $f_n \to f$, $f_n' \rightrightarrows f'$, then $\forall p$, $\frac{f_n(x) - f_n(p)}{x - p} \rightrightarrows \frac{f(x) - f(x)}{x - p}$

Theorem Let (f_n) diff, $f_n \to f$, $f_n' \rightrightarrows g$. $g = f'$

A *modulus of continuity* is $W : [o, \infty) \to [0, \infty]$ s.t $W(0) = \underset{x \to 0}{W(x)} = 0$. f has a modulus of continuity at p if $\exists \omega_p$ (modulus) s.t $d(x,p) < \delta \implies d(f(x), f(p)) < W(\delta)$. f is continuous at o iff f has a modulus of continuity at p. Uniform continuous if modulus exists that is independent of p

A collection $F \subset \mathcal{F}(X,Y)$ is **equicontinuous** if $\forall p \in X \;\; \exists \omega_p$ s.t $\forall f \in F$, ω_p (modulus) for f at p. Equivalently, if $\forall \varepsilon > 0 \;\; \exists \delta > 0$ s.t $d(x,p) \leq \delta \implies d(f(x), f(p)) < \varepsilon$.

Sequence converges pointwise to a function if for every $x \in D$ and $\varepsilon > 0$ $\exists N > 0$ s.t $n > N \implies |f_n(x) - f(x)| < \varepsilon$

Let $f_n = \cos(x/n)$. Fix $x \in [0, \pi]$. Since $cos(z)$ is continuous, for any $\varepsilon > 0 \exists \delta > 0$ s.t $z \in (0, \delta) \implies |cos(z) - 1| < \varepsilon$. So equivalently $\exists N$ s.t $n > N \implies x/n \in (0, \delta)$. Not a type of continuity, for seq.

x^2 cont with $\omega_p(\delta) = 2|p|\delta + \delta^2$

Theorem If $f_n \to f$ pointwise, (f_n) equicont, then f cont

$\overline{B_2(0)} \subseteq C^0(\mathbb{R})$ is not compact. Take $f_n \in B_2(0)$, $f_n(1/n) = 1$, limit to 0. f_n can't have uniform limit because $f_n \to 0$ not uniform, same with subsets. The problem is C^0 is that convergence is not just about size, but issues with continuity.

Uniform limit of f_n being Lip does not imply that that f_n has bounded lip constant, f_n can have arbitrarily, oscillating steps. But if $f_n \rightrightarrows f$ in $C^0(X, Y)$, then $\forall p$ $\exists \omega_p$ mod that holds at p for f, all f_n

Proof: Fix $p \in X$. Let $\omega_{p,n}$, ω_p be sharp mod at p for f_n, f. Define $\overline{\omega_p}(r) = \sup\{\omega_p(r), \omega_{p,n}(r)\}_{n \in \mathbb{N}}$. By construction, $d(x, p) < \delta \implies d(g(x), g(p)) < \overline{\omega_p}(\delta)$. WTS $\lim \overline{\omega_p}(r) = 0$. Fix $\varepsilon > 0$. Since $f_n \rightrightarrows f$ $\exists N$ s.t $n > N \implies d(f_n(x), f_n(p)) < d(f(x) + f(p) + \varepsilon$. WLOG take $\omega_{p,n}(r) \leq \omega_p(p) + \varepsilon$ $(n > N)$. Everything less than n going to 0.

$n^{-1}\sin(n^2 x) \rightrightarrows 0$, $f_n' = ncos(n^2 x)$ not bounded. $|n^{-1}(n^2(x+\delta) - n^{-1}\sin(n^2 x)| \leq \min(n\delta, 2n^{-2}) \leq 2\delta$

Theorem If $F \subseteq C^0$ compact, then $\exists M \in \mathbb{R}^+$ s.t $|f(x)| \leq M$, $f \in F$ equicont

Theorem (Arzela-Ascoli) Let K compact, $F \subseteq C^0(K, \mathbb{R})$. If F bounded in C^0 and F equicont, then any seq in F has conv subseq in C^0

Proof: Let (f_n) seq in F. Let ω_p unif mod for F. $\forall n \in N$ $\exists$ finite collection p_i, δ_i s.t $\omega_{p_i}(\delta_i) < n^{-1}$. $K \subseteq B_{\delta_i}(p_i)$. Take E_n set of all such p_i. Since E_n finite, $\cup E_n$ countable, $\exists A \subseteq \mathbb{N}$ infinite s.t $\lim_A f(p)$ conv $(p \in \cup E_n)$. WTS if f_n conv unif along A. Fix $\varepsilon > 0$. $E = E_n$, s.t $n^{-1} < \varepsilon$, so $\omega_p(\delta_p) \leq \varepsilon$ $(p \in E)$. For $x \in K$, $\exists p \in E$ s.t $x \in B_{\delta_i}(p)$. For $m, n \in A$,

$$|f_n(x) - f_m(x)| \leq |f_n(p) + f_m(p)| + |f_m(x) - f_m(p)| + |f_n(x) - f_m(p)| \leq 2\varepsilon + |f_n(p) - f_m(p)|$$

Theorem For $f : [a, b] \to \mathbb{R}$ cont, $\exists (P_n)$ polynomials s.t. $P_n \rightrightarrows f$

An **algebra** A is a collection of functions s.t. for $f, g \in A$, $f + g \in A$, $fg \in A$, $\lambda f \in A$.

A (function) collection F separates points if $\forall x \neq y$, $\exists f \in F$ s.t $f(x) \neq f(y)$

Theorem (Stone-Weirstrauss) K compact, $A \subseteq C^0(K)$ algebra. If A separates points, it's dense $(C^0(K))$

Proof: First note from HW, $\exists (P_n) \rightrightarrows |x|$. So taking $.5f + g + .5P_n(f - g)$, we have $\min, \max \in \overline{A}$. Take $g \in A$ and define $\overline{g} = (g - g(y))(g(x) - g(y))^{-1} \in A$, $\overline{g}(x) = 1$, $\overline{g}(b) = 0$, $f(x)\overline{g} + f(y)(1 - \overline{g}) \in A$. Fix $x \in K$, $\varepsilon > 0$. $\forall y \in K$, $\exists g_y$ s.t $g_y(x) = f(x)$,$g_y(y) = f(y)$, and cont. So $\exists \delta_y$ s.t $|f - g_y| < \varepsilon$ on $B_\delta(y)$. $K \subseteq \cup_{i=1}^n B_{\delta_i}(y_i)$. $\exists \overline{g}_x \in A$, $\overline{g}_x \in B_\varepsilon(\max_i g_{y_i})$. $\overline{g}_x(x) \in B_\varepsilon(f(x))$. $\overline{g}_x > f - 2\varepsilon$. This is from above. From below, similarly take $\{x_i\}$ s.t $f + 2\varepsilon > \min_i g_{x_i}$. $\exists g \in A$, $g \in B_\varepsilon(\min_i \overline{g}_{x_i})$, $\max g \in B_{3\varepsilon}(f)$

Peano's Lemma: let $F : \mathbb{R}^2 \to \mathbb{R}$ cont and consider $y : [0, T] \to \mathbb{R}$, $y(0) = a$, $y'(t) = F(t, y(t)$ on $[0, T]$. $\forall a \in \mathbb{R}$ $\exists T > 0$ s.t solution exists on the interval.

Idea: Since F cont, for $t \approx 0, y \approx a, y' \approx F(0, a)$. Control y' using Lipschitz. Create approx seq y_n where we expect limit to solve. Use compactness to show limit exists, and then show its the solution. Consider a physics application: start at a travel at speed M for time T. Still in $[0, T) \times B_\delta(a)$. Consider example where $f'' = -f$, $f(0) = 0$, and $f(0) = 1$. Then we can write $f' = g$ and then we have $g' = -f$. This gives

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} f' \\ g' \end{bmatrix} \implies$$

Abbreviations

cmpt - compact
cont - continuous
conv - converge(s/nt/nce)
cplt - complete
cpt - compact
incr - increasing
LHS - left hand side
lp - limit point
nbhd - neighborhood
part - partition
unif - uniform(ly)
WLOG - without loss of generality
WTS - want to show

Weekly Homework 1

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Math 531: Real Analysis I

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1: $\exists y \in \mathbb{R}^+$ **s.t** $y^3 = 3$

This will follow similarly to the proof in class of the supremum of $\{x \in \mathbb{Q} | x^2 < 2\}$. Here, we instead define the set $E = \{x \in \mathbb{R}^+ | x^3 < 3\}$. We know $\sup E$ exists[1] by the (Dedekind) completeness of $\mathbb{R}$. We want to show that $(\sup E)^3 = 3$.

(Case 1) Define arbitrary $\alpha > 1$ s.t $\alpha^3 < 3$. We will show α isn't an upper bound. Let $\delta = 3 - \alpha^3$, implying $\delta \in (0, 1)$. Fix $\epsilon > 0$ s.t $\epsilon < \delta/(9\alpha^2) < \delta/9$. Note that for $n \in \mathbb{N}/\{1\}$ $\alpha^n > \alpha$ and $\epsilon^n < \epsilon$. Then

$$(\alpha + \epsilon)^3 = \alpha^3 + \epsilon^3 + 3(\alpha\epsilon^2 + \alpha^2\epsilon) < \alpha^3 + \epsilon(1 + 6\alpha^2) < \alpha^3 + (7\delta)/9 < \alpha^3 + \delta = 3$$

Therefore, $\alpha + \epsilon$ is not an upper bound of E, so neither is α

(Case 2) Now define $\alpha \in (1, 2)$ s.t $\alpha^3 > 3$. We will show α is not the least upper bound of E. Let $\delta = \alpha^3 - 3$, so $\delta \in (0, 1)$. Fix $\epsilon > 0$ s.t $\epsilon < \delta/(6\alpha^2)$. Since $\epsilon^3 > 0$, $-\epsilon^2 > -\epsilon$, and $-\alpha > -\alpha^2$

$$(\alpha - \epsilon)^3 = \alpha^3 + \epsilon^3 - 3(\alpha\epsilon^2 + \alpha^2\epsilon) > \alpha^3 - 6\epsilon\alpha^2 > \alpha^3 - \delta = 3$$

Therefore, α can't be the least upper bound because $\alpha - \epsilon$ is an upper bound

As mentioned, we know $\sup E$ exists. Further, $\sup E \in \mathbb{R}^+$ from the cases above. Define $y = \sup E$. In both cases, we defined α arbitrarily, meaning that we can make the general statement that for $u \in \mathbb{R}$, if $u^3 < 3$ or $u^3 > 3$, then $u \neq \sup E$. By contraposition, $y^3 = (\sup E)^3 = 3$ ■.

2: S **is a totally ordered set, and** $E \subseteq S$**.** x **is the greatest element of** $E \implies x = \sup E$

We will show that if E has a greatest element, then it is the supremum

Since S is totally ordered and $E \subseteq S$, then E is totally ordered (if this wasn't the case, there would be an immediate contradiction for S being totally ordered). Let x be the greatest element of E. Then x must be an upper bound because $\forall y \in E, y \leq x$. Let $z < x$. Then there exists at least one element in E that is greater than z, so z is not an upper bound. Therefore, $x = \sup E$ by definition.

[1] This fact will be implicitly used/assumed in later problems with analogous sets

3: $x \in \mathbb{R}^+ \cup \{0\}$**.** $x \leq \epsilon$ **for any** $\epsilon > 0 \implies x = 0$

We will perform a proof by contradiction.

Let $x \in \mathbb{R}^+ \cup \{0\}$ s.t $x \leq \epsilon$ for any $\epsilon > 0$. Assume $x \neq 0$. Then $x \in \mathbb{R}^+$. Let $\epsilon = x/2$. Then $\epsilon > 0$ and $\epsilon < x$, a clear violation of the initial conditions for x. Therefore, $x = 0$.

4: For non-empty $A, B \subseteq \mathbb{R}$**, let** $A + B \equiv \{x + y | x \in A, y \in B\}$**.** $\sup(A + B) = \sup A + \sup B$

We want to show (WTS) $\sup(A + B) \leq \sup A + \sup B$ **and** $\sup(A + B) \geq \sup A + \sup B$

Let $A, B \subseteq \mathbb{R}$ be nonempty and $x \in (A + B)$. Then $x \leq \sup(A + B)$. For some $a \in A$ and $b \in B$, we can write $a + b = x$ (addition axiom for fields). Therefore $a \leq \sup(A + B) - b$. Since x is arbitrary, a can be any element of A, meaning $\sup(A + B) - b$ (the right hand side - RHS) is an upper bound for the set A. So the RHS is also less than or equal to the least upper bound of A and as a result

$$\sup A \leq \sup(A + B) - b \implies b \leq \sup(A + B) - \sup A$$

By a similar argument, the RHS of the second inequality above is an upper bound for the set B and is thereby less than or equal to the least upper bound of B, leading to $\sup A + \sup B \leq \sup(A + B)$

For $a \in A$, $a \leq \sup A$. Also, for $b \in B$, $b \leq \sup B$. Combining the two inequalities,
$a + b \leq \sup A + \sup B$. This inequality must hold for any choice of $a \in A$ and $b \in B$, meaning the RHS is an upper bound for the set $A + B$. Therefore, the RHS is less than or equal to the least upper bound, and $\sup(A + B) \leq \sup A + \sup B$.

We have now proven $\sup(A + B) \leq \sup A + \sup B$ and $\sup(A + B) \geq \sup A + \sup B$. In order for both of these to hold, $\sup(A + B) = \sup A + \sup B$.

5: If E **is a non-empty subset of an ordered set s.t** α and β **are (respectively) lower/upper bounds of** E**, then** $\alpha \leq \beta$

From #2, E is an ordered set. Since E is non-empty, consider any element $x \in E$. By definition of lower/upper bounds and the transitivity property inherent to ordered sets,

$$\alpha \leq x \ \underline{\textbf{and}}\ x \leq \beta \implies \alpha \leq \beta$$

6: $A \subseteq \mathbb{R}$ **be non-empty and bounded below with** $-A = \{-x | x \in A\} \implies \inf A = -\sup(-A)$

Since A is bounded below, $\inf A \in \mathbb{R}$. Let $\alpha = \inf A$. We WTS $\alpha = -\sup(-A)$.

By the propositions of ordered fields in Rudin, for any $x \in A$

$$\alpha \leq x \implies -x \leq -\alpha$$

since α is a lower bound of A. From the definition of the set $-A$, this means $-\alpha$ is an upper bound of $-A$. We will show that it is the least upper bound ($\sup(-A)$) using a proof by contradiction.

Assume $-\alpha$ is not the least upper bound of $-A$ ($-\alpha \neq \sup(-A)$). Then there exists $y \in \mathbb{R}$ s.t $y < -\alpha$ and y is an upper bound of $-A$. Then for any $a \in -A$

$$a \leq y < -\alpha \implies \alpha < -y \leq -a$$

By definition, $-a \in A$. Because there exists a lower bound greater than α, $\alpha \neq \inf A$, a contradiction. Thus, $-\alpha = \sup(-A)$ so $\inf A = -\sup(-A)$.

7: No order can be defined in the complex field that turns it into an ordered field

We will perform a proof by contradiction.

Assume there is an order, $<$, such that $\mathbb{C}$ is an ordered field. Consider first that $0, i \in \mathbb{C}$. Trivially, $i \neq 0$. For instance, (Rudin 1.16a) $i^2 = -1 \neq 0 = 0^2$. Further, $i^2 < 0$, which yields a contradiction (from contraposition) of Rudin's proposition 1.18d for ordered fields ($x \in \mathbb{C}, x \neq 0 \implies x^2 > 0$). Therefore, no order can be defined in the complex field that turns it into an ordered field

8: Suppose $z = a + bi, w = c + di$. Define $z < w$ if either $a < c$ or both $a = c$ and $b < d$. This turns the set of all complex numbers into an ordered set. Does this ordered set have the least-upper-bound property?

First, a prelude on notation: in order to deal with all the cases in this problem, one essentially has to let $b = d$ and $d = b$ in some instances, which is counterintuitive. This is because the problem is supposed to be set up such that w and z (and their componets) are both general and specific objects. Unfortunately, the only way to remedy this is to introduce *more* notation, which is obviously more cumbersome. My personal confusion outweighs the distaste for more objects, so before we begin the proof we will define more notation and essentially reformulate the problem.

Define the strictly general objects $x, y \in \mathbb{C}$ by

$$x = p_1 + p_2 i$$
$$y = q_1 + p_2 i$$

Let $x > y$ if **(T1)** $p_1 > q_1$ or **(T2)** both $p_1 = q_1$ and $p_2 > q_2$. Now we can define the applied but arbitrary (potential values depend on the case or sub-case) objects $w, z \in \mathbb{C}$ by

$$w = c + di$$
$$z = a + bi$$

This is an equivalent setup to Rudin #9, simply with some additional notation partitioning. Also, note that implicitly we know $p_1, p_2, q_1, q_2, c, d, a, b \in \mathbb{R}$. This is because any complex number can be written as the sum of a real number (constant) and i multiplied by a real number (coefficient).

We will complete the proof by showing that this setup leads to the two tenets of an ordered set for $\mathbb{C}$.

First, we WTS that $z = w$ or $z < w$ or $w < z$. Since $\mathbb{R}$ is an ordered set, $a = c$ or $a < c$ or $c < a$. If $a < c$ then $z < w$ by **T1** ($p_1 = c, q_1 = a$). If $c < a$ then $w < z$ also by **T1** ($p_1 = a$, $q_1 = c$). If $c = a$ and $d = b$ then $z = w$ by a simple substitution. If $c = a$ and $b < d$ then $z < w$ by **T2** ($p_2 = d, q_2 = b$). If $c = a$ and $d < b$ then $w < z$ also by **T2** ($p_2 = b, q_2 = d$). We have assessed all possibilities of w and z, and in all cases $z = w$ or $z < w$ or $w < z$.

Second, if an additional complex number $k = m + ni$ is defined, we WTS if $z < k$ and $k < w$ then $z < w$. If $z < k$ then by **T1** and **T2** either $a < m$ or both $a = m$ and $b < n$. Similarly, if $k < w$ then either $m < c$ or both $m = c$ and $n < d$. If we assume both conditional statements, this results in the following sub-cases. The transitivity endowed by $\mathbb{R}$ will be used repeatedly

(Case 1) Assume $a < m, m < c$. $a < m < c$, so $z < w$ by **T1** ($p_1 = c, q_1 = a$)

(Case 2) Assume $a < m, m = c, n < d$. $a < m = c$, so $z < w$ by **T1** ($p_1 = c, q_1 = a$)

(Case 3) Assume $a = m, b < n, m < c$. $m = a < c$, so $z < w$ by **T1** ($p_1 = c, q_1 = a$)

(Case 4) Assume $a = m, b < n, m = c, n < d$. $a = c$ and $b < n < d$, so $z < w$ by **T2** ($p_2 = d, q_1 = b$)

All cases of z, w, k have been assessed, so $z < k, k < w \implies z < w$

Therefore, since this setup establishes both desired properties of the definition, it turns $\mathbb{C}$ into an ordered set.

We will show that the setup does not uphold the least upper bound property using a proof by contradiction.

Assume $\mathbb{C}$ has the least upper bound property. Define $E \subseteq \mathbb{C}$ by $E = \{0 + ri | r \in \mathbb{R}\}$. This set is bounded above by any element in $\mathbb{R}^+$ by **T1** ($q_1 = 0 < p_1$ if p_1 is made positive), in addition to infinitely many complex numbers. So by the least upper bound property, $\sup E$ exists. For $a, b \in R$ let $\alpha = a + bi$ be the supremum ($\alpha \in \mathbb{C}$ since E is bounded above). We know if $a < 0$, then α is not an upper bound. So this results in two possibilities: $a > 0$ or $a = 0$.

If $a > 0$, let $c = a/2$ and $\beta = c + bi$. Then $\beta < \alpha$ by **T1** ($p_1 = a, q_1 = c$) and β is an upper bound on E, meaning α isn't the least upper bound (contradiction).

If $a = 0$, then let $d = 2b$ and $\beta = di$. Then $\beta \in E$ and $\alpha < \beta$ by **T2** ($p_2 = d, q_2 = b$), meaning α isn't an upper bound (contradiction).

These are the only two possibilities, meaning $\mathbb{C}$ cannot have the least upper bound property.

Weekly Homework 2

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Math 531: Real Analysis I

April 23, 2022

1: Show whether the following are open and/or closed (w.r.t $\mathbb{R}$ and $d(x, y) = |x - y|$)

(a) $\mathbb{Q}$ is **neither**. $\mathbb{Q}$ is not open because every neighborhood of a rational contains a real number. Formally, for any $\epsilon > 0$ and $x \in \mathbb{Q}$ by the Archemdian property there exists $n \in \mathbb{N}$ s.t $x\sqrt{2}/n \in B_\epsilon(x)$ (open ball[1]). $\mathbb{Q}$ is not closed because $\mathbb{Q}^c$ is not open by a similar argument (also see Rudin 1.20b).

(b) $A = (0, 1]$ is **neither**.A is not closed because $1 \in A$ but for any $\epsilon > 0$, $B_\epsilon(1) \cap A^c \neq \emptyset$ (e.g. $1 + .5\epsilon \in B_\epsilon(1), A^c$), so $B_\epsilon(1)$ is not a subset of A. A is not closed because 0 is a boundary point (since for any $x \in (0, 1)$, $0 + x \in A$ and $0 - x \in A^c$) but not in A (so ∂A not subset of A).

(c) $\emptyset$ is **clopen**. $\emptyset$ is open since $\emptyset^\circ = \emptyset$ (the set of all $x \in \mathbb{R}$ s.t $B_\epsilon(x) \subseteq \emptyset$ for some $\epsilon > 0$ is empty). Similarly, $\emptyset$ is closed since $\overline{\emptyset} = \emptyset$ (the set of all $x \in \mathbb{R}$ s.t $\forall r > 0$, $B_r(x) \cap E \neq \emptyset$ is empty).

(d) $\{0\}$ is **closed**. Similar to part b, $\{0\}$ is not open because for any $\epsilon > 0$, $B_\epsilon(0) \cap \mathbb{R}\backslash\{0\} \neq \emptyset$ (e.g. $0 + .5\epsilon \in B_\epsilon(0), \{0\}^c$), so $B_\epsilon(0)$ is not a subset of $\{0\}$. $\{0\}$ is closed because it's compliment is open because from part c, $\mathbb{R}$ is open as the compliment to $\emptyset$, so $\{0\}^c$ must be open ($\{x \in (-\infty, \infty)\}$ open $\implies \{x \in (-\infty, 0) \cup (0, \infty)\}$ open otherwise there's an immediate contradiction).

(e) $A = \{1, .5, 1/3, \dots\}$ is **neither**. A is not open because, for instance, any open ball around 1 will not be a subset of A because it contains infinitely many real numbers that are either irrational or greater .5, the next smallest element (e.g. $1/\sqrt{2} \in B_{.5}(1)$ but is not in A). A is not closed because $\{0\}$ is a limit point (for any $r > 0$, $\exists n \in \mathbb{N}$ s.t $1/n \in B_r(0), A$) but not in the set (direct violation of Rudin definition).

(f) $A = \{0, 1, .5, 1/3, \dots\}$ is **closed**. A is not open because the reasoning from part e) still holds. A is closed because it now contains 0, and $\partial A = \{0, 1\} \subseteq A$. It should be immediate that this is ∂A, but just in case here's a formal proof. $\partial A = \{0, 1\}$ because for any $a < 0$ or $a > 1$, there there's a ball centered at a that doesn't intersect A (e.g. consider a ball of radius $< |x|$ for $a < 0$),for $b \in (0, 1)$ there's a ball centered at b that doesn't intersect A^c, 0 is a boundary point because by the Archemedian property for $\epsilon < 0$ there exists $n \in \mathbb{N}$ s.t $-\epsilon < -1/n < 0 < 1/n < \epsilon$, and 1 is a boundary point because it will be in every open ball (so will intersect A) and by the Archemedian property for $\epsilon < 0$ there exists $n \in \mathbb{N}$ s.t $1 < 1 + 1/n < \epsilon$ (so will intersect A^c).

[1] for this instance and for now on, $B_r(x)$ denotes an open ball of length r centered at x

2: Classify open and closed sets w.r.t $\mathbb{R}^2$ and the Paris metric ($d(x,y) = ||x|| + ||y||$ and $d(x,x) = 0$)

For any point x 6= 0, the ball of radius ||x||/2 will contain x itself and nothing else. Thus x is an open ball.

For open balls centered the origin, the ball Br(0) of radius r in the Paris metric is equal to the same ball with respect to the usual metric.

A set is open if and only if it is a union of open balls. Therefore any set which does not contain the origin is open, and any set which contains the origin is open if and only if it also contains some open ball (with respect to the usual metric) around the origin.

A set is closed if and only if its complement is closed. Therefore any set which contains the origin is closed, and any set E which does not contain the origin is closed if and only if $\inf\{||x|||x \in E\} > 0$

3: For metric space (X, d), $x \in X$, and $A \subseteq X$, define $d(x, A) = \inf\{d(x, y)|y \in A\}$. For $A, B \subseteq X$ define $d(A, B) = \inf\{d(y, B)|y \in A\}$ Show or disprove that..

(a) for $E \subseteq X$, $x \in \overline{E}$ iff $d(x, E)$
$x \in \overline{E} \implies x \in E$ and/or $x \in E'$. Clearly, $x \in E \implies d(x, E) = 0$. So now consider the $x \in E'$ case. By rudin 2.20, any neighborhood of a limit point will contain infinitely many points of E, and if there was a positive distance between x and E this could not be the case (immediate contradiction between having infinite points for every neighborhood; you could take a ball with radius less than the distance). To show the other direction, proof by contradiction: assume $d(x, E) = 0$ and $x \notin \overline{E}$, meaning $x \in \overline{E}^c$ which is open, so there exists $B_\epsilon(x) \subseteq \overline{E}^c$. Let $y \in E$. By the construction of our ball, there must be positive distance between x and y since $\overline{E}^c$ won't contain any points of E, so $d(x, y) \geq 0$. Since y is arbitrary, this is a contradiction of the assumption that $d(x, E) = 0$.

(b) closed $A, B \subseteq X$ are disjoint iff $d(A, B) > 0$
Counter example: $A = \{n|n \in \mathbb{N}\}$ and $B = \{n + 2^{-n}|n \in \mathbb{N}\}$. A and B are closed by its complement being a countable union and are disjoint, but the the distance (2^{-n}) limits to 0.

4: For metric space (X, d) and disjoint closed sets $E, F \subseteq X$, there exist open sets $U, V \subseteq X$ s.t $E \subseteq U, F \subseteq V, \text{and}\, U \cap V = \emptyset$

We will show you can create open balls around both sets that do not intersect (using #3 definitions).

Since E and F are closed and disjoint, $\forall x \in E$ and $y \in F$, $d(x, y) > 0$, implying for each x and y there exists infinitely many balls (e.g. if $B_1(x)$ works, so does $B_{.9}(x)$, $B_{99}(x)$, and so on) that do not intersect the other set (i.e. infinitely many $B_r(x) \cap F, B_q(x) \cap E = \emptyset$). Intuitively, this means we can create two collections of balls that do not intersect but contain the relevant sets.

Formalizing this idea, for each $x \in E$, define $\epsilon_x > 0$ s.t $B_{\epsilon_x}(x) \cap F = \emptyset$. Now we have collection of ϵ_x, and call the infimum of this collection ϵ and define $U = \cup_{x\in E} B_{.5\epsilon}(x)$. Clearly, $E \subseteq U$ because any $x \in E$ will be in one of the balls. Similarly, for each $y \in F$, define $\delta_y > 0$ s.t $B_{\delta_y}(y) \cap E = \emptyset$ and all $b > \delta_y$ lead to $B_b(y) \cap E \neq \emptyset$. Let δ be the infimum of the collection of all δ_y and $V = \cup_{y\in F} B_{.5\delta}(y)$, so $F \subseteq V$. E and F are open because the union of open balls are open.

Now we WTS that $U \cap V = \emptyset$. It should be clear from the definitions this is the case, but we will rigorously show it. Define $e \in \partial E$ and $f \in \partial f$ such that if $e \neq e' \in \partial E$ and/or $f \neq f' \in \partial F$ then $d(e, f) \leq d(e', f')$. Note $0 < d(\cup_{x \in E} B_\epsilon(x), F), d(\cup_{y \in F} B_{.5\delta}(y), E)$ by construction (the balls are open and E, F are closed). Therefore, when we cut the radii of the balls in half (as U and V do), the distance is less than half of the shortest distance between E and F, or more formally $d(U, F), d(V, E) < .5d(e, f)$. Because U and F do not extend to the midpoint of the shortest distance between E and F (or beyond), they cannot intersect, so $U \cap V = \emptyset$.

5: If $k \geq 3, x \& y \in \mathbb{R}^k, ||x - y|| = d > 0,$ and $r > 0$, prove that..

First, consider this useful conceptualization to aid for all 3 parts. Consider two circles with radius r, with centers connected by a line segment with length d. If $2r > d$, this means the circles are not tangent to each other because otherwise the centers could be connected by a segment equivalent to the sum of their radii (i.e. $2r$). Moreover, this also means there is space between the circles because otherwise the line segment would be shorter than the sum of the radii. In this problem, r is not (necessarily) a radius, but thinking of it as one is helpful for why each of these cases ends up the way they do. Each case will start in 2-d space (i.e. with circles) and then will build up a generalization in a quasi-inductive fashion.

(a) if $2r > d$, there are infinitely many $z \in R^k$ s.t $||z - x|| = ||z - y|| = r$
First, we will offer an intuitive explanation, and then try to formalize it a bit by finding a general form for z.

Consider the conceptualization above. If $k = 3$, we can consider x and y spheres. Now consider another sphere z. If z intersects x and y, the intersection will be a circle, and more precisely will be distance r between both x and y. Since mutual the distance between z $(2r)$ is greater than the distance between x and y, this means there are several spheres z which could satisfy this property, because for any $z \in \mathbb{R}^3$ there are an arbitrary number of linear combinations that will result in an object equidistant from x and y. For instance, consider $r = 4 > .5d$. Then by extension, there can be a move to a greater value of r by modifying z, and as long as the change in "coordinates" is directly proportional to the previous ratios of $(x_1, y_1, \dots)$, we can continue creating new z further away that will obviously be greater than twice the distance between x and y. If we consider $k = 4$, this will be valid by what we have shown for spheres, and so on for $k \geq 3$.

For a formal proof, let $z = .5x + .5y + a$, where $a \cdot a = r^2 - (.5d)^2$. Note, we can make r arbitrarily large in the $2r > d$ case, so by construction there are several such zs that work. Then

$$||z - x||^2 = ||.5(y - x) + a||^2 = .25(y - x) \cdot (y - x) + a \cdot (y - x) + a \cdot a = .25d^2 + 2a \cdot (y - x) - .25d^2 + r^2$$

since $x \cdot x = ||x||^2$ and $||x - y|| = d$. Now we are left with $||z - x||^2 = r^2 + 2a \cdot (y - x)$. It's also easy to show that $||z - y||^2$ is the same thing. These can both simplified to simply r^2 if $a \cdot (y - x) = 0$, in which case we are done because $|| \cdot || \geq 0$, so squaring terms doesn't distort anything. From linear algebra, we know there are still an arbitrary number of combinations such that this works (along with the equation for $a \cdot a$). If you assume a solution exists, then for instance at $k = 3$ and $t > 1$ you have

$$a_1(x_1 - y_1) - a_2(x_2 - y_2) - a_3(x_3 - y_3) \implies ta_1(x_1 - y_1) - ta_2(x_2 - y_2) - ta_3(x_3 - y_3$$

Remembering the original construction of z, this means infinitely many z satisfy $||z - x|| = ||z - y|| = r$.

(b) if $2r = d$, there is only one $z \in \mathbb{R}^k$ s.t a) holds
Intuitively, we can again consider the conceptualization at the top. More formally, we know that z must be the exact midpoint between x and y otherwise the solution $(2r)$ will not be equal to d. This follows explicitly by the triangle inequality

$$d = ||x - y|| \leq ||x - z|| + ||z - y||$$

If $||z - x|| = ||z - y|| = r$ and the above holds with equality, by standard properties of real numbers there is only one such r where equality holds

(c) if $2r < d$, there is no $z \in \mathbb{R}^k$ s.t a) holds
Consider the triangle inequality above again. If $||z - x|| = ||z - y|| = r$ and $2r < d$, then

$$||x - y|| = d \geq 2r = ||x - z|| + ||z - y||$$

a contradiction for the triangle inequality which we know holds for norms. So no such z exists

(d) For $k = 2$, we know from the conceptualization at the top only 2 such z is possible that a) holds. For the rest, we get the same result. If $k = 1$, part a) doesn't hold for any z, but the rest are the same (one for b) and none for c)).

6: E' is closed. E and $\overline{E}$ have the same limit points, and E and E'..

(a) We WTS E' is closed because it's equal to its closure $\overline{E'}$. By definition, $x \in E' \implies x \in \overline{E'}$, so we will prove the other direction by contradiction: assume $\exists x \in \overline{E'}$ s.t $x \notin E'$. This x must satisfy $\forall r > 0, B_r(x) \cap E' \neq \emptyset$. So for each $B_r(x)$, $\exists y \in E', B_r(x)$. By Rudin 2.20, *any* neighborhood of y contains infinitely many points of E (i.e many $z \in E$ arbitrarily close to y s.t $z \neq x$). Since there's a y for each r, this implies for every $\epsilon > 0$ there is also some $z \in B_\epsilon(x)$ s.t $z \neq x, z \in E$. So by definition, x is a limit point of E, contradiction. So $x \in E' \iff x \in \overline{E'}$, meaning $E' = \overline{E'}$.

(b) We WTS $E' = \overline{E}'$. We will prove this by showing the general result that $A' \cup B' = (A \cup B)'$.

First, note that $A \subseteq B \implies A' \subseteq B'$. This is because if $x \in A \subseteq B$, then every neighborhood of x contains a point of $A \backslash \{x\}$ and thereby $B \backslash \{x\}$, so $x \in B'$. Next, note that this implies $A' \cup B' \subseteq (A \cup B)'$ because it results in $A' \subseteq (A \cup B)'$ and $B' \subseteq (A \cup B)'$. Finally, we will show $(A \cup B)' \subseteq A' \cup B'$. Let $x \notin A' \cup B'$. Then there exists neighborhoods U and V of x s.t $U \cap (A \backslash \{x\}), V \cap (B \backslash \{x\}) = \emptyset$. So $M = U \cap V$ is a neighborhood of x s.t $M \cap \big((A \cup B) \backslash \{x\}\big) = \emptyset$. Therefore, $x \notin (A \cup B)'$, implying $(A \cup B)' \subseteq A' \cup B'$. Combining results, $A' \cup B' = (A \cup B)'$.

Since $\overline{E} = E \cup E'$, $\overline{E}' = E' \cup (E')'$. However, since E' is closed from a), we know $(E')' \subseteq E'$. Therefore, $\overline{E}' = E'$

(c) As alluded to in b), E' and E do not always have the same limit points. In fact, from a) we know that the limit points of E' are a subset of E', or $(E')' \subseteq E'$. For example, if $E' = \{1\}$, $(E')' = \emptyset$.

7: For $i \in \mathbb{N}$, let $\{A_i\}$ be subsets of a metric space

(a) $(n = 1, 2, \dots)$ $B_n \cup_{i=1}^{n} A_i \implies \overline{B}_n = \cup_{i=1}^{n} \overline{A_i}$
We will prove this by showing the general result that $\overline{A \cup B} = \overline{A} \cup \overline{B}$

First note that $A \subseteq B \implies \overline{A} \subseteq \overline{B}$ because it also implies that $A \subseteq \overline{B}$, and from standard topology texts the closure of A can be thought of as the intersection of all closed sets containing it, so since B is closed $\overline{A} \subseteq \overline{B}$. This means $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$, so $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. And in a similar argument leading to the first result, since $A \cup B \subseteq \overline{A} \cup \overline{B}$, where the RHS (right hand side) is closed, then $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Therefore, combining results yields equality.

Now if we consider adding A_i' as a union to each A_i, this will be equivalent to the union of closures by the result above. Further, if we consider $\overline{B} = B \cup B'$, if we assume it does not contain and $x \in A_i$ (for some i) we will immediately arrive at a contradiction. Therefore, $B' = \cup_i A'$, and so therefore $\overline{B}_n = \cup_{i=1}^{n} \overline{A_i}$

(b) $B = \cup_{i=1}^{\infty} A_i \implies \cup_{i=1}^{\infty} \overline{A}_i \subset \overline{B}$
This follows directly from $A \subseteq B \implies \overline{A} \subseteq \overline{B}$ above. We have

$$(\forall i)\ A_i \subset B = \cup_{i=1}^{\infty} A_i \implies (\forall i)\ \bar{A}_i \subset \bar{B} \implies \cup_{i=1}^{\infty} \bar{A}_i \subset \bar{B}$$

(c) Example

$$(0,1) = \cup_{i \in \mathbb{N}} (1/n, 1) \implies (0,1) = \cup_{i \in \mathbb{N}} \overline{(1/n, 1)} \subset [0,1] = \overline{(0,1)}$$

8: $x \in E \subseteq R^2$ open $\implies x \in E'$? If $E \subseteq \mathbb{R}^2$ closed?

(a) Since E is open, let $x \in E, E^\circ$. Thus, for $\epsilon > 0$ we have $B_\epsilon(x) \subseteq E$. However, because E is open, we know that $\exists y \in B_\epsilon(x)$ s.t $y \neq x$. The intuition on this comes from some of the previous examples; if x were singleton or discrete it wouldn't be open, therefore there must be an accumulation of points. Therefore, x is a limit point by definition.

(b)
This does not hold for closed sets. There are a myriad of counter examples from the fact that finite sets in $\mathbb{R}^2$ are closed from Rudin, namely a set with one element $\{(0,1)\}$ will not be a limit point because there is no other element of the set, so there does not exist $y \neq x$ but $y \in E$ for any neighborhood of $x = \{(0,1)\}$.

9: For a set E..

(a) E° is open
Let $x \in E^\circ$. Then for some $\epsilon > 0$, $B_\epsilon(x) \subseteq E$. This implies for infinitely many $y \in B_\epsilon(x)$, and (for each) there exists $r \in (0, \epsilon)$ s.t $B_r(y) \subseteq E$. This means each $y \in B_\epsilon(x)$ is an interior point, so by definition $B_\epsilon(x) \subseteq E^\circ$. So to recap: $\forall x \in E^\circ\ \exists \epsilon > 0$ s.t $B_\epsilon(x) \subseteq E^\circ$

(c) $G \subset E$ open $\implies G \subset E^\circ$
If $G \subset E$ is open, then $x \in G \implies x \in G^\circ$. This means that $x \in E^\circ$ also because given $\epsilon > 0$, $B_\epsilon(x) \subseteq G \subset E$. To clarify: we have shown $x \in G \implies x \in G^\circ \implies x \in E^\circ \implies G = G^\circ \subset E^\circ$

(e) $E^\circ = \overline{E}^\circ$?
No. Consider $E = (-1,1)\backslash\{0\}$, which we have implicitly shown is open in previous problems (and in class). Therefore, $E^\circ = E$. $\overline{E} = [-1,1]$, as it is clearly the smallest closed set containing E (Rudin). The interior of this set is the closure less its boundary, so $\overline{E}^\circ = (-1,1) \neq E^\circ$

(f) $\overline{E} = \overline{E^\circ}$?
No. From #1, consider $E = \{0\}$, which we know is closed so $\overline{E} = \{0\}$. However, $E^\circ = \emptyset$ because each ball around 0 will obviously contain something that is not 0, so it can't be a subset of 0. Because the empty set is closed, its closure is itself, so $\overline{E^\circ} = \emptyset \neq \overline{E}$.

10: For $x, y \in \mathbb{R}$, are the following metrics?

(a) $d_1(x,y) = (x-y)^2$
No, this isn't a metric because it doesn't preserve the triangle inequality. Consider that
$d(3,1) = 4 > d(3,2) + d(2,1) = 2$

(b) $d_2(x,y) = \sqrt{|x-y|}$
Yes. If $x \neq y$, by the addition axioms $x - y \neq 0 \implies |x-y| \neq 0 \implies \sqrt{|x-y|} \neq 0$. If $x = y$, by the same logic $d_2(x,y) = 0$. That shows both directions of positive definite. $|x-y| = |y-x|$, so the square-roots are also the same, showing symmetry. The triangle inequality holds because we already know that it holds for the absolute value and the absolute value is ≥ 0, in which case square-root preserves order, meaning

$$|x-y| \leq |x-z| + |y-z| \implies \sqrt{|x-y|} \leq \sqrt{|x-z|} + \sqrt{|y-z|}$$

(c) $d_3(x,y) = |x^2 - y^2|$

No, this doesn't preserve positive definite, for example $d_3(-1,1) = 0$

(d) $d_4(x,y) = |x-2y|$
No, this doesn't preserve positive definite, for example $d_3(1,.5) = 0$

Weekly Homework 3

Paul B.
Math 531: Real Analysis I

April 23, 2022

1: If (nonempty) $E \subseteq X$ closed and $K \subseteq X$ compact, E, K disjoint $\iff d(E, K) > 0$ (HW# 2 def)

($\Longleftarrow$) Proof by contradiction: assume $E \cap K \neq \emptyset$ and let $x \in E \cap K$. From the definition of the greatest lower bound, $d(E, K) > 0 \implies \forall e' \in E$ and $k' \in E$ $d(e', k') > 0$, but by definition $x \in E, K$ and if we let $e' = x$ and $k' = x$ then $d(e', k') = d(x, x) = 0$, contradiction.

($\Longrightarrow$) Proof by contradiction: assume $d(E, K) = 0$. Because E, K are disjoint, this intuitively means E contains a subset of elements that are arbitrarily close to K, and likewise K contains a subset arbitrarily close to E. More formally, for any $\epsilon > 0$, define the sets

$$E_\epsilon = \{e \in E | d(e, K) < \epsilon\} \ \underline{\textbf{and}} \ K_\epsilon = \{k \in K | d(E, k) < \epsilon\}$$

By construction, $E_\epsilon, K_\epsilon \neq \emptyset$ for any ϵ (otherwise the infimum of the distance between E and K would not be 0 as assumed). Now consider an ordered, natural numbers indexing of both sets. Impose that these representations are both infinite (elements can repeat if necessary). Explicitly,

$$E_{\epsilon,n} = \{e_i \in E_\epsilon | d(e_i, K) \geq d(e_{i+1}, K)\}_{i \in \mathbb{N}} \ \underline{\textbf{and}} \ K_{\epsilon,n} = \{k_i \in K_\epsilon | i \in \mathbb{N} : d(E, k_i) \geq d(E, k_{i+1})\}_{i \in \mathbb{N}}$$

Therefore, because ϵ is arbitrary, for $e_i \in E_{\epsilon,n}$ and $k_i \in K_{\epsilon,n}$, we must have $\lim_{i \to \infty} d(e_i, k_i) = 0$. Because K is compact, any infinite subset must have at least one limit point in K. So let $\alpha \in K'_{.5\epsilon,n}, K$. Next, we will prove the result that $\exists j \in \mathbb{N}$ s.t $e_j \in B_\epsilon(\alpha)$, which will show $\alpha \in E'$.

$\alpha \in K'_{.5\epsilon,n} \implies B_{.5\epsilon}(\alpha)$ contains infinitely many elements of $K_{.5\epsilon,n}$ (Rudin 2.20). More precisely, there any infinitely many $k_i \in B_{.5\epsilon}(\alpha), K_{.5\epsilon,n}$. Since $\lim_{i \to \infty} d(e_i, k_i) = 0$, for any $\epsilon > 0$ $\exists j \in \mathbb{N}$ s.t $d(e_j, k_j) < .5\epsilon$. Because we know there exists such a $k_j \in B_{.5\epsilon}(\alpha), K_{.5\epsilon,n}$, this implies $e_j \in B_\epsilon(\alpha)$ by the triangle inequality ($d(\alpha, e_j) \leq d(\alpha, k_j) + d(e_j, k_j) \leq \epsilon$). So $\alpha \in E'$ by definition (ϵ is arbitrary, $\alpha \neq e_j$ otherwise contradiction is immediate). $\alpha \in E$ since closed sets contain their limit points. However, $\alpha \in K$ also, so $E \cap K \neq \emptyset$, contradiction.

2: Let (X, d) be limit point compact: for every infinite $E \subseteq X$, $E' \neq \emptyset$

(a) For any $\delta > 0$ $\exists \{x_i\}_{i=1}^N (N \in \mathbb{N})$ s.t $X \subseteq \cup_{i\in[1,N]} B_\delta(x_i)$
Proof by contradiction: assume X is not totally bounded. Then from the definition given in the probelm, $\exists \epsilon > 0$ s.t $x \in X \implies X \not\subseteq B_\delta(x)$. By the definition of a subset, this implies $\exists y \in X$ s.t $y \notin B_\epsilon(x)$. $B_\epsilon(x) \cup B_\epsilon(y)$ could be made into its own ball for some $z \in X$ that does not contain some other element in the set. This process could continue infinitely, making a union and reforming a ball that does not include some element in X since it's not totally bounded. Let $\delta > 0$ be the minimum of all the radii that would be formed in the process. We also can define a collection through iteration $A = \{x_i \in X | x_{i+1} \notin \cup_{j=1}^i B_\delta(x_j)\}_{i\in\mathbb{N}}$. Then $x_i, x_j \in A (i \neq j) \implies d(x_i, x_j) \geq \delta$. $A \subseteq X$ (and is infinite), so by definition of limit point compact there exists $\alpha \in A', X$. By the definition of limit point, $\exists x_i, x_j \in B_{.5\delta}(\alpha)$ s.t $x_i \neq x_j$ since a ball around α contains infinitely many elements of A. By the triangle inequality, $d(x_i, x_j) < \delta$, contradiction.

(b) $\{F_n\}_{n\in\mathbb{N}}$ nested sequence of nonempty closed subsets of $X \implies \cap_{n\in\mathbb{N}} F_n$ nonempty
Let $I = \{0\} \cup \mathbb{N}$. Construct a sequence $A = \{f_n\}_{n\in I}$ s.t $f_n \in F_n$ and $f_i \notin F_n (i < n)$, which is possible by the definition of nested. So informally, this is a set consisting of one point from each F_n. Clearly A is an infinite subset of X, so by the definition of limit point compact $\exists \alpha \in A', X$. Additionally, closed sets contain their limit points, and F_0 is closed with $f_n \in E (n \in I) \implies f_n \in A$, so $A \subseteq F_0$ and thus F_0 must contain points of A'. Trivially note $\cup_{i\in\mathbb{N}} F_i \subseteq F_0$ by nestedness. Again by Rudin 2.20, $B_\epsilon(\alpha)$ contains infinitely many elements of A $(\epsilon > 0)$. This means that we can take finitely many points out of A and this will still hold. So now we will exploit this fact to show that $\alpha \in \cup_{n=0}^\infty F_n$ using an inductive argument. By construction, $A \backslash \{f_0\} \subseteq F_1$. F_1 is closed, meaning it must contain its limit points. α will still be a limit point for $A \backslash \{f_0\}$ because the arbitrary ball alluded to earlier contains infinitely many points of $A \backslash \{f_0\}$, meaning it will obviously satisfy the definition[1] of a limit point for this new set. This means we can apply the same argument that yielded $\alpha \in F_0$ to show that $\alpha \in F_1$. We can see from here we will arrive at a classic inductive result:

$$A \backslash \{f_0, f_1, \ldots, f)n\} \subseteq F_{n+1}$$

and $\alpha \in F_{n+1}$ since the same arguments we used for F_1 apply. Therefore, we can generalize this result infinitely, giving us $\alpha \in \cup_{n=0}^\infty F_n$, obviously meaning that this infinite intersection is nonempty.

(c) (X, d) is compact
From a), for $\delta = 1$, $\exists \{x_i\}_{i=1}^N \in X$ s.t $X \subseteq \cup_{i\in[1,N]} B_1(x_i)$ (with $N \in \mathbb{N}$). Therefore, we can construct $\{G_\alpha\}$ an open cover for X. Now, we will consider a proof by contradiction based on this established premise. This will follow a similar strategy to the proof for proving $[a, b]$ is compact.

Assume X is not compact. Then $\exists A = B_1(x_n)$ which cannot be covered with a finite set. If we partition A into closed balls, then one of these balls must also not have some possible finite cover (by our non-compactness assumption, otherwise contradiction). Formally, consider a ball partitioning using radii $d = (1, x_n)/2$. Select the ball that is "finitely uncoverable". Consider this first partitioning process yielding A_1, and the next A_2, and so on. Then we have $A_n \subseteq A_{n+1}$ and from class A_n has radius $2^{-n} d(x_n, 1)$. Because this setup fulfils the assumptions/conditions needed for the result in b), we can say that $\exists \beta \in \cap_{i\in\mathbb{N}} A_i$ s.t $\beta \in A'$ (and is not $\emptyset$). Referencing the open cover developed at the beginning, this implies $\exists a$ s.t $\beta \in G_a$. By the openness of G_a, $\exists \epsilon > 0$ s.t $B_\epsilon(\beta) \subseteq G_a$. Because β is a limit point of A, by the radii properties above (also from the

[1] s.t $x \in E'$ iff $\forall r > 0$, $\exists y \in B_r(x)$ where $y \neq x, y \in E$

aforementioned proof in class) $A_n \subset B_{2^{1-n}d(x_n,1)}(\beta)$. Because we definite A_n such that the radii are shrinking towards 0, by the Archimedian property $\exists m \in \mathbb{N}$ such that the radius of A_m is less than ϵ. Let the radius of A_m be r. Then $A_m \subseteq B_\epsilon(\beta) \subseteq G_a$. However, E_m is supposed to not have a finite cover by construction, contradiction.

3: For X infinite and $p, q \in X$ let $d(p, q) = 0$ if $p = q$ and 0 otherwise.

(a) Prove this is a metric
$d(p, q) = 0 \implies p = q$ by the construction of the metric. The implied by direction follows by simple contraposition: $d(p, q) \neq 0 \implies d(p, q) = 1 \implies p \neq q$. Therefore we have proven positive definiteness.

If $p = q$, then $d(p, q) = d(p, p) = d(q, p) = 0$. If $p \neq q$, then $d(p, q) = d(q, p) = 1$. Therefore we have shown symmetry.

For $p, q, s \in X$, $d(p, s) + d(q, s) \geq 0$. If they all are equal, $0 = 0$. If $p = q \neq s$, then $0 = d(p, q) \leq d(p, s) + d(p, s)$. If $s = p \neq q$, then $d(p, q) = 1 = d(p, s) + d(q, s)$. If $s \neq p \neq q$, then $d(p, q) = 1 < 2 = d(p, s) + d(q, s)$. Therefore, we have shown the triangle inequality in all cases.

Therefore, d is a metric.

(b) Which subsets are open, closed, and compact
Open sets: singleton sets are open because a ball with radius less than 1 will be a subset of the singleton. Singleton sets are their closures, so their complement is open, meaning $X \backslash \{x\}$ is also open. Balls with radius greater than one are also open. This also means that the entire space X is open. The empty set will still naturally subset itself so its open.

Closed sets: As alluded to above, singleton sets are closed. However, since they are also open, by the compliment property $X \backslash \{x\}$ is also closed. $\emptyset$ is its own closure, so its also closed. The entire space is also closed since its compliment ($\emptyset$) is open.

Compact sets: Finite subsets of X are compact. This will include the singletons, for example.

4: $K = \{1/n | n \in \mathbb{N}\} \cup \{0\}$ is compact

Let $\{G_\alpha\}_{\alpha \in A}$ be an open cover of K with $\mathbb{G} = \cup_{\alpha \in A} G_\alpha$. We will prove that there must be a finite subcover $\{G_\beta\}_{\beta \in B}$ (where $B \subseteq A$ is finite).

First, we will prove that $\exists a \in A$ s.t for some $N \in \mathbb{N}$, $[0, 1/N] \subseteq G_a$. $\{0\}$ must be in some G_a, otherwise $K \not\subseteq \mathbb{G}$ and thus $\{G_\alpha\}_{\alpha \in A}$ won't be an open cover. So $\exists a \in A$ s.t $\{0\} \in G_a$. Because G_a is open, there must exist $r > 0$ s.t $B_r(0) \subseteq G_a$ by the definition of an open set. So $G_a \cap K \backslash \{0\} \neq \emptyset$ and $\exists x > 0$ s.t $x \in B_r(0)$. From a result from class, there exists $N \in \mathbb{N}$ s.t $.5x \in (1/N, N)$. So $\{1/N\} \in B_r(0)$ and thus $[0, 1/N] \subseteq B_r(0) \subseteq G_a$. By a similar line of argumentation, $\exists b \in A$ s.t for some $M \in \mathbb{N}$, $[1/M, 1] \subseteq G_b$. WLOG, let $M \leq N$.

In constructing our open cover, so far we have G_a and G_b. Now we need to find an finite indexing that will include $(1/N, 1/M)$. Note that there are finitely many natural numbers between M and N. So now consider the set $C_n = \{n \in \mathbb{N} | n \in (M, N)\}$. For each $i \in C_n$, there exists at least one $c \in A$ s.t $1/i \in G_c$. WLOG, for each i let c_i be the smallest value that this holds, and thus we can form the set $C = \{c_i\}_{i=M+1}^{N-1}$. Now, the set $B = \{a\} \cup \{b\} \cup C$ is finite, $B \subseteq A$, and $\{G_\beta\}_{\beta \in B}$ coveres K. Therefore, we can develop a finite subcover, $\{G_\beta\}_{\beta \in B}$, for every open cover of K, so K is compact.

5: Give an open cover of $(0, 1)$ with no finite subcover

Let $K = (0, 1)$. Let $G_n = (1/n, 1)$. Then $\cup_{i=1}^{\infty} G_i = (0, 1)$, so its an open cover for K (open covers don't need to be a proper subset). Let $B \subseteq \mathbb{N}$ be finite. Then
$\exists N$ s.t $n \leq N$ and $1/N \leq 1/n \;\forall n \in B \implies (1/N, 1) \subset (1/n, 1)$. This implies that
$\cup_{\beta \in B} G_\beta = (1/N, 1)$. So any potential finite subcover you could construct will always have uncovered space between 0 and some $1/N$.

6: If A, B are disjoint closed sets in a metric space X

(a) prove they are separated
Since A and B are closed, $\overline{A} = A$ and $\overline{B} = B$. This means

$$\overline{A} \cup B = A \cup \overline{B} = A \cup B = \emptyset$$

by the definition of disjoint. So also by definition, A and B are separated.

(b) prove they are separated if A, B are instead disjoint and open.
Proof by contradiction: assume A and B are not separated. Then $\overline{A} \cap B$ or $A \cap \overline{B}$ are not equal to $\emptyset$. However, because open sets are equal to their interiors and A and B are disjoint and open, $A^\circ \cap B^\circ = \emptyset$. $\partial A = \overline{A} \backslash A$ (since the interior is equal to its set). So consider the case where $\partial A \cap B \neq \emptyset$ (which must be the case if $\overline{A} \cap B$ by the parameters of this problem). Then $\exists x \in A^\circ, B^\circ$ (since the sets are equal to their interiors and by the definition of boundary point). This is an immediate contradiction for $A^\circ \cap B^\circ = \emptyset$. A similar contradiction will arise in the only other case of $A \cap \partial B \neq \emptyset$. Therefore, the sets must be separated.

7: Are closures and interiors of connected sets always connected?

We will prove closures of connected sets are connected by contradiction. Assume E is connected but $\overline{E}$ isn't. Then for $E = A \cup B$, $A \cap \overline{B}$ or $\overline{A} \cap B$ is not the empty set. I proved in the last homework a result that implies $\overline{E} = \overline{A} \cup \overline{B}$. To not get confusing let $\mathbb{A} = \overline{A}$ and $\mathbb{B} = \overline{B}$. We showed in homework 2 that the a set and its closure have the same limit points, so $\overline{\mathbb{A}} = \mathbb{A} \cup \mathbb{A}' = \mathbb{A} \cup A' = \mathbb{A}$. So we have now shown that the closure of a closure is just the closure itself. This implies by the separation of $\overline{E}$

$$\overline{\mathbb{A}} \cap \mathbb{B} = \mathbb{A} \cap \overline{\mathbb{B}} = \overline{A} \cap \overline{B} = \emptyset$$

By the last equality, since $X \subset \overline{X}$, this implies that $A \cap \overline{B} = \overline{A} \cap B = \emptyset$, contradiction.

Interiors are not always connected. Consider the counterexample[2] where we are in $\mathbb{R}^2$ and A is the unit disk centered at the origin and B is the interior of the unit disk centered at $(0, 2)$. Then $E = A \cup B$ is connected because $A \cap \overline{B}$ will be nonempty (consider $(0, 1)$). However, the interior of E, which is the union of the interiors of A and B, is not connected because the aforementioned intersection point is no longer contained in $\overline{B}^\circ$, so $A^\circ \cap \overline{B}^\circ = \overline{A}^\circ \cap B^\circ = \emptyset$

[2] credit to Sarah

Weekly Homework 4

Paul B.
Math 531: Real Analysis I

April 23, 2022

1: Prove that convergence of $\{s_n\}$ implies convergence of $\{||s_n||\}$. Is the converse true?

For clarity, define the relevant sequences with respect to metric space (X, d). Also, we will use a corollary to the triangle inequality that for $x, y \in X$ $d(||x||, ||y||) \leq d(x, y)$ (see Rudin p.88).

Say that $\{s_n\}$ converges to $p \in X$. Fix $\epsilon > 0$. Then by the definition of convergence $\exists N \in \mathbb{N}$ s.t $\forall n > N$ $d(s_n, p) < \epsilon$. By the result above, $n > N \implies d(||s_n||, ||p||) \leq d(s_n, p) < \epsilon$. Therefore, by definition $\{|s_n|\}$ is a convergent sequence.

The converse is not true. As a counterexample, $\{||(-1)^{n-1}||\}$ converges to 1 but $\{(-1)^{n-1}\}$ is not a convergent sequence.

2: Consider the following limits

For a),b), c), and potentially future homeworks, we will first prove a version of the *continuous mapping theorem*[1]: Suppose $f : X \to X$ is continuous and $x_n \to x$. If $x, x_n \in X \ \forall n$, $f(x_n) \to f(x)$
Proof: (Rudin 4.6 implies it is clear, but just to be safe) Fix $\epsilon > 0$. With respect to a metric space (X, d), by the definition of continuity $\exists \delta > 0$ s.t $d(x_n, x) < \delta \implies d(f(x_n), f(x)) < \epsilon$. So by the definition of convergence, $\exists n > N$ s.t $x_n \in B_\delta(x)$, meaning $d(f(x_n), f(x)) < \epsilon$.

(a) $\lim(\sqrt{n^2+n} - n) = .5$
First we will simplify the sequence to make it easier to work with

$$\sqrt{n^2+n} - n = \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n} = \frac{n}{\sqrt{n^2+n} + n} = \left(\frac{\sqrt{n^2+n}}{n} + 1\right)^{-1}$$

Consider that $\frac{\sqrt{n^2+n}}{n} = \frac{\sqrt{n}\sqrt{n+1}}{n} = \frac{\sqrt{n+1}}{\sqrt{n}} = \frac{\sqrt{n}\sqrt{1+1/n}}{\sqrt{n}} = \sqrt{1+1/n}$. Therefore we have shown

$$\sqrt{n^2+n} - n = \frac{1}{\sqrt{1+1/n} + 1}$$

By the result proven above[2], $\sqrt{1+1/n} \to 1$, so $\sqrt{n^2+n} - n = \frac{1}{\sqrt{1+1/n}+1} \to .5$

[1]Please scroll to the appendix for a proof of $a_n \to a \implies \sqrt{a_n} \to \sqrt{a}$ that does not directly use continuity

[2]also trivially, from class, $\frac{1}{n} \to 0$. The CMT result was used because in this case we have $\sqrt{1+1/n} + 1 \to 2$, and $g(x) = 1/x$ is a continuous mapping because all possible inputs ($\sqrt{1+1/n} + 1$ with $n \in \mathbb{N}$) are greater than 1

(b) For $\alpha, \beta \in \mathbb{R}$, $\lim(\alpha + n)/(\beta + n) = 1$
We will first show that $|\alpha/n|, |\beta/n| \to 0$. The key here is that α, β are fixed. So it follows the same as $1/n \to 0$. But to be explicit, fix $\epsilon > 0$. WLOG let $|\beta| \leq |\alpha|$. For all $n > |\alpha|/\epsilon$,
$|\beta/n| < |\alpha/n| < |\alpha/\frac{|\alpha|}{\epsilon}| = |\alpha|/\frac{|\alpha|}{\epsilon} = \epsilon$. Thus, $\alpha/n, \beta, n \to 0$. Now note that

$$\frac{\alpha + n}{\beta + n} = \frac{\frac{\alpha}{n} + 1}{\frac{\beta}{n} + 1} \to 1$$

explicitly by the CMT-type result at the top of #2 (in case there is a concern with inverses).

(c) For $r_0 = \sqrt{2}$, define (a_n) by $r_n = \sqrt{2 + r_{n-1}}$. Show it converges to a limit ≤ 2
We will prove that the sequence is bounded by 2 and monotonistic with induction, and then invoke the monotone convergence theorem to show it has a limit, which must be ≤ 2.

We WTS that $r_n < 2$. We will prove this by induction. For the base case of $n = 1$:

$$r_1 = \sqrt{2 + r_0} = \sqrt{2 + \sqrt{2}} < \sqrt{2 + 2} = \sqrt{4} = 2$$

Now assume $r_n < 2$. For the inductive step we will show $r_{n+1} < 2$

$$r_{n+1} = \sqrt{2 + r_n} < \sqrt{2 + 2} = 2$$

Also, note that by construction $r_n > 0$. So r_n is a bounded sequence. Further, we will prove r_n is monotonistic by induction. Formally, we WTS that $r_n > r_{n-1}$. For the base case of $n = 1$

$$r_1 = \sqrt{2 + \sqrt{2}} > \sqrt{2 + 0} = r_0$$

Now assume $r_n > r_{n-1}$. For the inductive step we will show $r_{n+1} > r_n$ by

$$r_{n+1} = \sqrt{2 + r_n} > \sqrt{2 + r_{n-1}} = r_n$$

Now we have shown (r_n) is a bounded, monotonistic sequence. So by the monotonistic convergence theorem it converges. For a brief proof by contradiction: assume its limit is greater than 2. Call the limit α. Fix $\epsilon > 0$. Then $\exists N$ s.t $\forall n > N$ $d(r_n, \alpha) < \epsilon$. But because $0 < r_n < 2 < \alpha$ $\forall n$, it follows that $d(r_n, 2) < d(r_n, \alpha) < \epsilon$. So 2 is a limit of r_n. But $r_n \neq \alpha$, and the limits of convergent sequences are unique, contradiction. Thereofre, the limit of $(a_n) \leq 2$

3: Let (X, d) be a metric space, (a_n) a sequence in X, and E the set of (x_n)'s subsequential limits

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First, a general note on this problem with clarification from Dr. Stokols: when a limit is infinite, we consider the sequence divergent, so $\pm\infty$ cannot be in a set of subsequential limits.

(a) (a_n) bounded $\implies$ E bounded
Consider first the trivial case where (a_n) has no subsequential limits, so $E = \emptyset$ (this could happen in some space that is not $\mathbb{R}^n$, for instance). Then E is bounded because $\emptyset$ is bounded.

Now consider the E nonempty case. We will perform a proof by contradiction: assume that E is unbounded. Because (a_n) is bounded,from class we know $\exists R \in \mathbb{R}^+$ s.t $\forall n$ $||a_n|| \in [-R, R]$. So let $M = 2R$. Then $\forall m, n \in \mathbb{N}$ $d(a_n, a_m) < M$. Pick $\alpha \in E$ such that α is the limit of the subsequence $(a_n)_{n \in A}$, where we are using the notation from class that $A = \{n_k | k \in \mathbb{N}\} \subseteq \mathbb{N}$, and for some $j \in \mathbb{N}$ $d(a_j, \alpha) > 4M$. This is possible because (a_n) is bounded but E isn't, so we can pick an element of

E that is arbitrarily large. Fix $\epsilon \in (0, M)$. By the definition of a subsequential limit, $\exists N$ s.t for each $i \in A$ that is greater than N, $d(a_i, \alpha) < \epsilon$. Recalling that $\forall m, n \in \mathbb{N}$ $d(a_n, a_m) < M$, combining results and using the triangle inequality yields

$$0 < 4M < d(\alpha, a_j) \leq d(\alpha, a_i) + d(a_i, a_j) < M + \epsilon < 2M$$

Since $4M > 2M$, this is a contradiction. We have analyzed all possible cases of E, so it's bounded.

(b) E bounded $\not\Rightarrow$ (a_n) bounded
Consider the infinite sequence $\frac{1}{2}, 2, \frac{1}{3}, 3, \ldots$. Per the note at the top and confirmed in office hours for this specific sequence, this sequence has only one subsequntial limit, 0, so $E = \{0\}$ is bounded. However, the sequence is unbounded because it has a subsequence that approaches infinity.

4: For a set X and $d : X \times X \to \{0, 1\}$ the discrete metric, (X, d) is complete

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Per instruction, we will be focusing on Cauchy completeness.

Let (a_n) be a Cauchy sequence in (X, d). Fix $\epsilon \in (0, 1)$. Then by the definition of Cauchy, $\exists N \in \mathbb{R}$ s.t $\forall j, k > N$ $d(a_j, a_k) < \epsilon$. Since $d(a_j, a_k) \in \{0, 1\}$ and $0 \leq d(a_j, a_k) < \epsilon < 1$, these two can only jointly hold if $d(a_j, a_k) = 0$. Therefore by the definition of the discrete metric, $\forall j, k > N$, $a_j = a_k$. This means the terms in any Cauchy sequences in (X, d) must become constant at some point and are thus convergent. Formally: still using N based on the (arbitrarily fixed) ϵ, $\forall j, k > N$, let $c = a_j = a_k$. Then for all $j > N$ $|a_j - c| < \epsilon$. So (a_n) is by definition a convergent sequence in (X, d) because $c \in X$ by construction, meaning every Cauchy sequence is convergent in (X, d).

5: Let (x_n) be convergent in (X, d). A rearrangement $y_n = x_{g(n)}$ converges to the same limit

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First, let $g : \mathbb{N} \to \mathbb{N}$ be some bijection. Call $y_n = x_{g(n)}$ the corresponding *rearrangement*. We WTS that the sequence of rearranged terms (y_n) converges to same limit as (x_n). Note: it is important to be careful about the definition of $g(\cdot)$. Per the definition of a rearrangement, g maps the position of an item in the rearranged sequence to the corresponding position in x. Therefore, $g^{-1}(k)$ gives the position of the kth item in the (x_n) sequence in (y_n) (bijection inverses are well-defined).

The idea behind this proof is that if one extends far enough out in a sequence, then its rearrangement also must be past a certain point. Moreover, for each $M \in \mathbb{N}$, $\exists N \in \mathbb{N}$ s.t $\forall n > M$ $g(n) > N$, meaning that given an N that satisfies the definition of convergence for (x_n), we can find a natural number for (y_n) that satisfies the definition for the same limit. To these ends, define the following sets for a given $M \in \mathbb{N}$

$$A_M = \{n \in \mathbb{N} | i \in [1, M]\} \ \underline{\textbf{and}} \ B_M = \{g^{-1}(a) | a \in A\}$$

B_M gives all the positions of the first M terms of (x_n) in the rearranged sequence. So for all n larger than the largest element of B_M (which corresponds to an index position), y_n will not be one of the first M elements of (x_n). Now we will set out formally showing this result using precise indexing.

Let α be the limit of (x_n). Fix $\epsilon > 0$. Then by definition of a convergent sequence $\exists M \in \mathbb{N}$ s.t $\forall n > M$ $d(a_n, \alpha) < \epsilon$. This implies there are finitely many terms where the distance between α is greater than epsilon, in fact at most M of them. The definitions of A_M and B_M above imply $\delta = \max B$ exists because A_M, and therefore B_M, are finite, bounded sets. Let $N = \max\{M, \delta\}$. Then by construction, $\forall n > N$, $y_n \notin (x_n)_{n \in A}$. In other words, for all $n > N$, we have guaranteed any y_n cannot be found in the first M terms of (x_n) because $g(n) > M$. More formally: $\forall n > N\ d(y_n, \alpha) = d(x_{g(n)}, \alpha) < \epsilon$. Therefore y_n converges to α by definition.

6: Let (X, d) be complete and define $f : X \to X$ s.t $\exists \lambda \in [0, 1)$ s.t
$\forall x, y \in X$, $d(f(x), f(y)) \leq d(x, y)$. Then $\exists x_c \in X$ s.t $f(x_c) = x_c$

With respect to f, we will introduce some useful notation: define $f^n(x) = f(f(\dots f(x)))$ to be the n nested iterations of $f(f(x))$ (i.e. using the previous value of $f(x)$ as the input n times). Subsequently define $\{x_n\}_{n\in\mathbb{N}}$ s.t given $x_0 \in X$ $x_n = f^n(x_0)$. We WTS[3] that $\exists x_c \in X$ s.t $f(x_c) = x_c$.

By the contraction property of f given in the problem, for some $\lambda \in [0, 1)$

$$d(x_3, x_2) = d(f(x_2), f(x_1) \leq \lambda d(x_2, x_1) = \leq \lambda^2 d(x_1, x_0) \implies d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0)$$

since we can just reapply the property on the left hand side as many times as we want to get the right hand side. Let $m > n$. Thus, by applying the triangle inequality an amount of times with respect to the difference between m and n and again using the contraction property of f

$$\begin{aligned} d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) &\leq \Big(\sum_{i=n}^{n-1} \lambda^i\Big) d(x_1, x_0) \\ &= \lambda_n \Big(\sum_{i=0}^{m-n-1} \lambda^i\Big) d(x_1, x_0) \\ &\leq \frac{\lambda^n}{1-\lambda} d(x_1, x_0) \end{aligned}$$

from the hint given in the problem. Note $\frac{\lambda^n}{1-\lambda} \to 0$. Fix $\epsilon > 0$. Because $d(x_1, x_m)$ is constant, $\exists N$ s.t $\forall n > N$ $\frac{\lambda^n}{1-\lambda} d(x_1, x_0) < \epsilon$. Therefore $\{x_n\}$ is Cauchy, and since we're in a complete metric space, define its limit to be x_c. By applying the triangle inequality and contraction property

$$d(f(x_c), x_c) \leq d(f(x_c), f^n(x_0)) + d(x_c, f^n(x_0)) \leq \lambda d(x_c, f^{n-1}(x_0)) + d(x_c, f^n(x_0))$$

By the definition of a point of convergence, $\exists N' \in N$ s.t $\forall n > (N' + 1)$ $d(x_c, x_n) < \epsilon/(\lambda + 1)$ (because we already fixed epsilon earlier). So let $M = \max\{N, N' + 1\}$. Then for $n > M$

$$d(f(x_c), x_c) \leq \lambda d(x_c, f^{n-1}(x_0)) + d(x_c, f^n(x_0)) = \lambda d(x_c, x_{n-1}) + d(x_c, x_n) < \epsilon$$

Because ϵ is arbitrary, we can conclude $f(x_c) = x_c$

[3] Greg and I have seen this notation and result in a previous Macro class

7: Define $(x_n^m) \in \mathbb{R}$ s.t $|x_n^m| \leq 1\ \forall m, n \in \mathbb{N}$. $\exists A \subseteq \mathbb{N}$ infinite s.t $(x_n^m)_{n\in A}$ is convergent $(\forall m \in \mathbb{N})$

Since each $(x_n^m)_n$ is bounded, each one has a convergent subsequence by Bolzano-Weirstrass. Moreover, every subsequence of $(x_n^m)_n$ is also bounded with a convergent subsequence.

Take $A_0 \subseteq \mathbb{N}$ corresponding to a convergent subsequence of $(x_n^0)_{n\in\mathbb{N}}$ and let $a_0 = \inf A_0$, which is an element of A_0 since every subset of $\mathbb{N}$ has a least element. Take $A_1 \subseteq A_0$ corresponding to a convergent subsequence of $(x_n^1)_{n\in A_0}$ and let $a_1 = \inf A_1 \cap \{n > a_1\}$. Then $a_1 > a_0$ and $a_1 \in A_1$. Continue this process iteratively, so $A_m \subseteq A_{m-1}$ corresponds to a convergent subsequence of $(x_n^m)_{n\in A_{m-1}}$ and let $a_m = \inf A_m \cap \{n > a_{m-1}\}$. Then $a_m > a_{m-1}$ and $a_m \in A_m$.

Because A_n are nested, and any subsequence of a convergent subsequence also converges to the same limit, $(x_n^m)_{n\in A_k}$ is convergent for $k \geq m$. We cannot take the union over the natural numbers because "A_∞" might be empty. So instead let $A = \{a_k | k \in\}$. Note that a_k is an increasing infinite sequence, so A corresponds to a subsequence. Further, for $k \geq m$, $a_k \in A_k \subseteq A_m$. So $A \subseteq A_m \cup \{a_0, \ldots, a_{m-1}\}$. Since $(x_n^m)_n$ converges along A_m, and adding finitely many terms does not affect converges, $(x_n^m)_n$ converges along A. So A gives us what we want.

Appendix We will prove the special case that $a_n \to a \implies \sqrt{a_n} \to \sqrt{a}$
We will break up the proof into two cases. Initially, assume $a \neq 0$. Then

$$0 \leq |\sqrt{a_n} - \sqrt{a}| + \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \leq \frac{|a_n - a|}{\sqrt{a}} \to 0$$

so $\sqrt{a_n} \to a$ by the squeeze theorem from calculus. If $a = 0$, then for each $\epsilon > 0\ \exists N > 0$ s.t for $n > N$, $|a_n - 0| < \epsilon^2 \implies |\sqrt{a_n} - 0| < \epsilon$ ■.

Weekly Homework 5

Paul B.
Math 531: Real Analysis I

April 23, 2022

1: Let $a < b \in \mathbb{R}$ and $(x_n) \in [a, b]$. $\liminf x_n = b \implies \lim x_n = b$

We will first define some additional notation to add onto what Dr. Stokols introduced in class. For the set $E_N = \{x_i | i \geq N\}$, let $\alpha_N = \sup E_N$ and $\beta_N = \inf E_N$. Trivially note from the Rudin definitions this means for all $n \in \mathbb{N}$ $\alpha_n \geq \beta_n$. Also, from the problem, notice that $b = \liminf x_n = \lim \beta_n$ (and similarly $\limsup x_n = \lim \alpha_n$).

We will show that $\limsup x_n = b = \liminf x_n$, which we will prove implies the sequence limits to b.

We know $E_N \subset [a, b]$ $\forall N \in \mathbb{N}$. $\sup[a, b]$, which is b since[1] closed, bounded sets contain their supremum, must be $\geq$ than the sup of any subset, so $(\forall N)$ $\alpha_N \leq b \implies b - \alpha_N \geq 0$, so

$$b - \beta_N \geq b - \alpha_N \geq 0 \implies \beta_N - b \leq \alpha_N - b \leq 0 \implies |\alpha_N - b| \leq |\beta_N - b|$$

because $a_N \geq \beta_N$. Fix $\epsilon > 0$. Since $b = \lim \beta_n$, there exists $M \in \mathbb{N}$ s.t $\forall n > M$

$$|\alpha_n - b| \leq |\beta_n - b| \leq \epsilon$$

Thus, $b = \lim \alpha_N = \limsup x_n$. Now consider the original sequence (x_n). Recall the aforementioned $M \in \mathbb{N}$ we defined conditional on a fixed arbitrary ϵ. We know by construction/definition that $\beta_n \leq x_n \leq \alpha_n$, but we also know from results derived above that for $n > M$

$$b - \epsilon < \beta_n \leq x_n \leq \alpha_n < b + \epsilon \implies |x_n - b| < \epsilon$$

So by definition $\lim x_n = b$

2: Let $f : \mathbb{R}^* \to \mathbb{R}$ with at least one fixed point $(c \in \mathbb{R})$. Define $a_{n+1} = f(a_n)$

(a) a_n does not necessarily limit to a fixed point
Define $f : \mathbb{R}^* \to \mathbb{R}$ s.t $f(x) = x^2$ if $x \in \mathbb{R}$ and $f(x) = -2$ if $x \in \{\infty, -\infty\}$. $f(0) = 0$ and $f(1) = 1$, so $f(\cdot)$ has two fixed points. If we start the sequence with $a_0 = 2$, the sequence $a_{n+1} = f(a_n)$ will clearly diverge to ∞. It cannot limit to a fixed point because this definition of a fixed point implies it must lie in the codomain, which is $\mathbb{R}$, ∞ cannot be a fixed point, especially with how we've defined the function[2], so a_n does not limit to a fixed point.

[1] Dr. Stokols clarified it was okay to use these properties in the HW
[2] note that at ∞ the sequence would start over again along a path towards divergence

(b) If c is the only fixed point of f and $\limsup f(a_n) = f(\limsup a_n)$ (same for $\liminf$), $a_n \to c$
Note that the following result was proven in #1 (also see Rudin 3.18c)
For a sequence (x_n), if $\exists c \in \mathbb{R}$ s.t $c = \limsup x_n = \liminf x_n \implies x_n$ is convergent and $\lim x_n = c$

We will first manipulate $\limsup f(a_n)$ and $\liminf f(a_n)$ to yield fixed points, which we know there are only one of, so therefore $\limsup f(a_n) = \liminf f(a_n)$, and we can apply the result above to show the sequence converges to this fixed point.

Because $a_n = f(a_{n-1})$, $\limsup f(a_n) = f(\limsup a_n) = f\big(\limsup f(a_{n-1})\big)$. Trivially, we know $\limsup f(a_n) = \limsup f(a_{n-1})$. Explicitly, this is because by the definition of a limit, if we let $a = \limsup f(a_n)$, for any $\epsilon > 0$ $\exists N$ s.t $|\lim\big(\sup\{f(a_n)|n > N\}\big) - a| < \epsilon$, so by extension $N+1$ works for showing $\limsup f(a_{n-1}) = a$. With this definition of a,
$\limsup f(a_n) = f\big(\limsup f(a_{n-1})\big) \implies a = f(a)$. Therefore, a is a fixed point. Similarly for $\liminf$, if we let $b = \liminf f(a_n)$, then we know by the same given property of f that $\liminf f(a_n) = f\big(\liminf f(a_{n-1})\big)$, which implies that $b = f(b)$. So b is a fixed point. Since we know c is the only fixed point, $c = a = b$, so therefore $\limsup f(a_n) = \liminf f(a_n) = c$. We know from the result above that this implies $f(a_n)$ is convergent to c. Because $f(a_n) = a_{n+1}$, $a_{n+1} \to c$. We can apply the argument showing $\limsup f(a_n) = \limsup f(a_{n-1})$ to confirm $a_n \to c$.

3: $\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n$ ($(a_n), (b_n) \in \mathbb{R}$, $\limsup a_n + \limsup b_n \not\to \infty - \infty$)

Define $A = \limsup a_n$ and $B = \limsup b_n$

First we will take case of the real case: assume $-\infty < A, B < \infty$. Fix $\varepsilon > 0$. Then by the definition of the lim sup $\exists N_1, N_2 \in \mathbb{N}$ s.t

$$(\forall n > N_1)\ a_N < A + \varepsilon/2 \ \underline{\textbf{and}}\ (\forall n > N_2) b_n < B + \varepsilon/2$$

Let $N = \max\{N_1, N_2\}$. Then $\forall n > N$ $a_n + b_n < A + B + \varepsilon$. Therefore[3]
$\limsup(a_n + b_n) \le A + B + \varepsilon$. Because ε is arbitrary, this implies that $\limsup(a_n + b_n) \le A + B$. But just to be rigorous: suppose that were not the case for a contradiction. Then $\limsup(a_n + b_n) - (A + B) > 0$, so by the Archimedian property there exists $\varepsilon > 0$ s.t $\limsup(a_n + b_n) - (A + B) > \varepsilon$, contradiction. So given our definition of $A + B$, $\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n$.

Now we have two cases left to consider: $\infty = A = B$ and $-\infty = A = B$. The first case is trivial because in $\mathbb{R}^*$ its not possible for ∞ to be dominated from above. So assume $A = B = -\infty$. If we show that this implies $\limsup(a_n + b_n) = -\infty$, we are done. Suppose this is not the case. Let $\limsup(a_n + b_n) = C$. Either $C \in \mathbb{R}$ or C is ∞. However, we will show neither of these options are possible because A and B are diverging towards $-\infty$. Define $A_N = \{a_i | i \ge N\}$ and $B_N\{b_i | i \ge N\}$. For any $n > N$, we know that $a_n \le \sup A_N$ and $b_n \le \sup B_N$. Because $\sup A_N, \sup B_N \xrightarrow{N} -\infty$, this implies that for any for any $D \in \mathbb{R}$, there exists $M \in \mathbb{N}$ s.t $\forall n > M \in \mathbb{N}$ s.t $a_n, b_n < -|D|$. Taking the sum of sequence terms shows that no real number (and by extension ∞) can bound $a_n + b_n$ from below. So the least upper bound of the set $\{a_i + b_i | i \ge N\}$ must be diverging towards $-\infty$ as $N \to \infty$, thus $C = -\infty$. In case you are not convinced of this argument, consult the even more rigorous proof found in the appendix.

[3]Implied by Rudin 3.18c and Rudin 3.19. $A + B$ is just a value in $\mathbb{R}$. By Rudin 3.18c its limsup would be the limit of a sequence of repeating terms of $A + B$, which is just $A + B$. Then Rudin 3.19 gives the final result

4: Given the following sequences, do their infinite sums converge or diverge..

(a) $a_n = \sqrt{n+1} - \sqrt{n}$
$\sum a_n$ **diverges**. This is because $\sum_{i=1}^{N} = \sqrt{N-1} - 1$ because each term in the sequence will eliminate the term before it (e.g $\sum_{i=1}^{2} = \sqrt{3} - \sqrt{2} + \sqrt{2} - 1 = \sqrt{3} - \sqrt{1}$). So we know that the partial sums $s_n = \sqrt{n-1} - 1$. Clearly $s_n \to \infty$, so the infinite sum diverges

(b) $b_n = a_n/n$
$\sum b_n$ **converges**. Note that

$$b_n = \frac{a_n(\sqrt{n+1} + \sqrt{n})}{n(\sqrt{n+1} - \sqrt{n})} = \frac{1}{n(\sqrt{n+1} - \sqrt{n})} < \frac{1}{n(2\sqrt{n})} < \frac{1}{n^{1.5}}$$

In class, we established that $\sum 1/n^p$ converges if $p > 1$. So $\sum b_n$ converges by the comparison test

5: If $a_n \geq 0$, $\sum a_n$ converges $\implies \sum \sqrt{a_n}/n$ converges

We will prove this using the fact that $x_1, x_2 \geq 0 \implies 2(\sqrt{x_1 x_2}) \leq x_1 + x_2$, which follows from Cauchy-Schwarz and is otherwise known as the AM-GM inequality. Using this result

$$\frac{2\sqrt{a_n}}{n} = \frac{2\sqrt{a_n}}{\sqrt{n^2}} \leq a_n + 1/n^2$$

with $x_1 = a_n$ and $x_2 = 1/n^2$, which are both non-negative. We know $1/n^2$ converges from #4 b), so the RHS converges since $\sum a_n$ converges. Thus $\sqrt{a_n}/n$ converges by comparison test.

Appendix - #3, $A = B = -\infty$ case

We have shown $C = \limsup(a_n + b_n) \neq \infty$. So we have one sub-case left to consider. Define $C_N = \{a_i + b_i | i \geq N\}$. Assume $C \in \mathbb{R}$. By construction of C_N, this implies that given $n > N \in \mathbb{R}$, $a_n + b_n \leq \sup C_N$. Because $C \in \mathbb{R}$, let N' be the smallest positive integer such that $\sup C_{N'}$ is not ∞. Let $C' = \sup C_{N'}$. Because lim sup is decreasing and $C \in \mathbb{R}$, note $C' \geq \sup C_n$ and $\sup C_n \in \mathbb{R}$ for all $n > N'$. By the definition of the least upper bound, given $M \in \mathbb{N}$ s.t $M > N'$, $\exists m > M$ s.t $\forall \varepsilon > 0$, $\sup C_M - \varepsilon < a_m + b_m$ (if this did not hold then there would be a smaller upper bound). Combining the past couple results, by the Archemedian property $\exists a \in \mathbb{N}$ s.t $-|a \cdot C'| \leq a_n + b_n$ for all $n > N'$. Because A and B are diverging towards negative infinity, we know there exists $n \in \mathbb{N}$ s.t $a_n, b_n < -|D|$. $-|a \cdot C'| \in \mathbb{R}$, contradiction.

Weekly Homework 6

Paul B.
Math 531: Real Analysis I

April 23, 2022

1: $\sum a_n$ converges and (b_n) monotonic and bounded, then $\sum a_n b_n$ converges

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Note per instruction, $(a_n) \in \mathbb{C}$ and $(b_n) \in \mathbb{R}$. Also, note that by the monotonic convergence theorem (b_n) is convergent, so define $b \in \mathbb{R}$ s.t $b_n \to b$. Define S_n as $\sum_{i=0}^{n} a_i$ (partial sums). Because $\sum a_n$ is convergent, we know $\exists M \in \mathbb{R}^+$ s.t $|S_n| \leq M \ \forall n$. For this proof, we will use Rudin 3.41 and mimic the steps of the proof for Rudin 3.42. We will also use two cases in order to satisfy the conditions/proof strategy of 3.42. One could just apply 3.42 directly but we will be rigorous.

(Case 1) Assume b_n is monotonic increasing
Define $c_n = b - b_n$. (c_n) is a decreasing sequence since $-b_n \geq -b_{n+1} \implies b - b_n \geq b - b_{n+1}$ by the usual properties of $\mathbb{R}$ and the fact that (b_n) is increasing. Fix $\varepsilon > 0$. Because we also know (c_n) converges to 0 and is monotonic decreasing, it must follow that $c_n \geq c_{n+1} \geq 0 \ \forall n$, otherwise contradiction for either monotonistic decreasing or converging to 0. So putting all this together, because $c_n \to 0$, $\exists N$ s.t $c_N \leq \frac{\varepsilon}{2M}$. Let $N \leq p \leq q$. By (c_n) decreasing, $c_p \leq c_N$, so by Rudin 3.41

$$|\sum_{n=p}^{q} a_n c_n| = |\sum_{n=p}^{q-1} S_n(c_n - c_{n+1}) + S_q c_q - S_{p-1} c_p| \leq M|\sum_{n=p}^{q-1}(c_n - c_{n+1}) + c_q - c_p| = 2Mc_p \leq 2Mc_N < \varepsilon$$

where we also used the fact that c_p is non-negative. Note that $|\sum_{n=p}^{q} a_n c_n|$ is equivalent to the absolute difference in the partial sums of $\sum a_n c_n$ for $N \leq p \leq q$. Therefore, the partial sums are Cauchy, thus convergent, so $\sum a_n c_n$ converges. Now, we have

$$\sum a_n c_n = \sum a_n b - \sum a_n b_n \implies \sum a_n b_n = b \sum a_n - \sum a_n c_n$$

$b \sum a_n$ is convergent because $\sum a_n$ is convergent. So from a result in class, $\sum a_n b_n$ is the sum of two convergent series, so is therefore convergent itself.

(Case 2) Assume b_n is monotonic decreasing
Define $c_n = b_n - b$. We have all the same conditions we established for the sequence used in Case 1 by the same logic: (c_n) is a decreasing sequence because $b_n \geq b_{n+1} \implies b_n - b \geq b_{n+1} - b$. We also know (c_n) converges to 0 and $c_n \geq c_{n+1} \geq 0$. Therefore, by repeating the steps in Case 1, $\sum a_n c_n$ is convergent. So $\sum a_n b_n = \sum a_n c_n - b \sum a_n$ is the sum of two convergent series, and is convergent.

We have dealt with all cases. So $\sum a_n b_n$ converges.

2: If $a_n \geq 0$ and $a_n \to 0$, then $\forall \varepsilon > 0$ there exists a subsequence of (a_n) s.t $\sum_{k=0}^{\infty} |a_{n_k}| < \varepsilon$

This proof will follow similarly to the strategy used to show that the set of all subsequential limits is closed. Of course, we will adapt the strategy for the series setting as needed. Essentially, we will construct a subsequence ourselves that leads to an easy creasing of an infinite, convergent series that bounds from above.

First, note that $|a_n| = a_n \forall n$ since the sequence is non-negative. Fix $\varepsilon > 0$. Now define the following sub-sequence, let a_{n_0} be an element of (a_n) s.t $a_n < \varepsilon/2$. Now, set $n_k > n_{k-1}$ and define a_{n_k} s.t $a_{n_k} < \frac{\varepsilon}{2^{k+1}}$. Such a construction is possible because $a_n \to 0$, so there are infinitely many sequence terms that are less than any positive real number. Now, note that we have

$$\sum_{k=0}^{\infty} |a_{n_k}| = \sum_{k=0}^{\infty} a_{n_k} < \frac{\varepsilon}{2}(\sum_{k=0}^{\infty} \frac{1}{2^k}) = \varepsilon$$

by the standard geometric sum property.

3: $\liminf \frac{a_{n+1}}{a_n} > 1$ and $\forall n > N$ $\frac{a_{n+1}}{a_n} \geq 1$ (some N) imply divergence. What's the connection?

If the first condition holds, then the second condition also holds. The converse is not true.

Define $\alpha = \liminf a_{n+1}/a_n > 1$ and $E_M = \{a_{i+i}/a_i | i \geq M\}$. Fix $\varepsilon > 0$. By the definition[1] of lim inf, $\exists N$ s.t $\forall n > N-1$ $|\inf E_n - \alpha| \leq \varepsilon$. This implies that $\inf E_N > 0$ since $N > N-1$ (if $\inf E_N \leq 0$ then the previous result would not hold for any $\varepsilon < 1$). We can extend this logic behind this argument to say that we know $\inf E_N \geq 1$, but we will be a bit more precise about this point. Suppose $1 > \inf E_N$ for a contradiction. Then since $\alpha > 1 > \inf E_N > 0$

$$1 - \inf E_N = |\inf E_N - 1| \leq |\inf E_N - \alpha| < \varepsilon$$

so by definition $\liminf a_n = 1$, but limits must be unique, contradiction. Thus, by our construction of E_N and intuitively because $(\inf E_n)$ is an increasing sequence, $1 \leq \inf E_N \leq a_{n+1}/a_n$ $\forall n > N$.

However, if the second condition holds, the first does not necessarily hold. Consider $a_n = 2$ $\forall n$. $\sum a_n$ diverges by $a_{n+1}/a_n \geq 1$ $\forall n > 1$. But $\liminf a_{n+1}/a_n = 1$, so the ratio test is ambiguous.

4: The comparison test can encourage some unhelpful intuition, so in light of this..

Note form Dr. Stokols: all terms of every series are strictly positive

(a) Given $\sum x_n$ divergent, $\exists (a_n)$ s.t $\limsup \frac{a_n}{x_n} = 0$ and $\sum a_n$ diverges
Let $S_n = \sum_{i=0}^{n} x_i$ (partial sums). Then let $a_n = \frac{x_n}{S_n}$. We will first show that a divergent series of only positive terms must be diverging to ∞. This will prove that $\limsup a_n/x_n$ is ∞. We will then use the strategy from Rudin #11 to show that $\sum a_n$ diverges.

First, note that S_n does not converge to anything because $\sum x_n$ is a divergent series. Also, since $x_n > 0$, $S_n < S_n + x_{n+1} = S_{n+1}$ $\forall n$. Thus, S_n is a monotonic increasing sequence. Assume for a contradiction that S_n is bounded. Then by the monotonic convergence theorem, S_n has a limit, contradiction. Therefore, S_n is unbounded. It follows trivially that the lim sup of a positive,

[1] using the in class definition. See last homework for a bit more context if needed

increasing, and unbounded sequence will be infinite. Therefore $\limsup \frac{a_n}{x_n} = \limsup \frac{1}{S_n} = 0$. Now, define S'_n as the partial sums of $\sum a_n$. If $m > n$ and $k = m - n$, since S_n is strictly increasing

$$|S'_m - S'_n| = \frac{x_{n+1}}{S_{n+1}} + \cdots + \frac{x_{n+k}}{S_{n+k}} \geq \frac{\sum_{i=1}^{k} x_{n+i}}{S_{n+k}} = \frac{S_{n+k} - S_n}{S_{n+k}} = 1 - \frac{S_n}{S_{n+k}} > 0$$

because the positive, strict monotonicity conditions yield $\frac{S_n}{S_{n+k}} \in (0, 1)$. Therefore, the sequence of partial sums of $\sum a_n$ is not Cauchy, so now convergent, meaning $\sum a_n$ diverges

(b) Given $\sum y_n$ soncvergent, $\exists (b_n)$ s.t $\liminf \frac{b_n}{y_n} = \infty$ and $\sum b_n$ converges
Let $r_n = \sum_{i=n}^{\infty} y_n$ and $b_n = \frac{y_n}{\sqrt{r_n}}$. We will first show that $r_n \to 0$, which will show $\liminf \frac{y_n}{b_n} = \infty$. Then, we will use the strategy outlined in Rudin #12 to show that $\sum b_n$ is convergent.

Because y_n is convergent, its partial sums (S_n) are Cauchy. Therefore, for any $\varepsilon > 0$, $\exists N$ s.t $\forall m, n > N$ $|S_m - S_n| < \sum_{i=n}^{m} y_i < \varepsilon$ (since $y_i > 0\ \forall i$). Because this holds $\forall m, n$ given an N, we can represent this in the context of a limit definition as $|\sum_{i=n}^{\infty} y_i - 0| = r_n < \epsilon$. Thus, $r_n \to 0$. From the result I proved in the appendix of HW # 5, this implies $\sqrt{r_n} \to 0$. Moreover, r_n is a decreasing sequence because y_n is a positive sequence, so $r_{n+1} < r_{n+1} + y_n = r_n$. Therefore, for any $M \in \mathbb{R}$, $\exists N$ s.t $\forall n > N$ $\frac{1}{\sqrt{r_i}} > M$. Thus, we now have $\liminf \frac{b_n}{y_n} = \liminf \frac{1}{r_n} = \infty$ because the limit of the greatest lower bound for $\{\frac{1}{\sqrt{r_i}} | i \geq n\}$ is infinite.

Recall from above $0 < r_{n+1} < r_n = r_{n+1} + y_n$. Then $\frac{r_{n+1}}{r_n} \in (0, 1) \implies 1 + \sqrt{\frac{r_{n+1}}{r_n}} < 2$ and further

$$\implies \frac{a_n}{\sqrt{r_n}} = \frac{1}{\sqrt{r_n}}(r_n - r_{n+1}) = \sqrt{r_n} + \sqrt{r_{n+1}} - \left(\sqrt{r_{n+1}} + \frac{r_{n+1}}{\sqrt{r_n}}\right) = \left(1 + \sqrt{\frac{r_{n+1}}{r_n}}\right)\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$$
$$\implies \frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

Now taking this over sums, since $\sqrt{r_n} \to 0$ (and implicit from above $b_n > 0\ \forall n$)

$$0 < \sum_{i=0}^{n} b_i = \sum_{i=0}^{n} \frac{a_i}{\sqrt{r_i}} < 2\sum_{i=0}^{n}(\sqrt{r_i} - \sqrt{r_{i+1}}) = 2(\sqrt{r_0} - \sqrt{r_{n+1}}) \to 2\sqrt{r_0}$$

So $\sum b_n$ converges by comparison because $0 < r_0 = \sum y_n$, a convergent series.

5: Given $\sum a_n$ with partial sums S_N, if $C[a_n] = \lim_N (\sum_{i=0}^{N} S_i)(N+1)^{-1}$, called the Cesàro sum, converges, then $\sum a_n$ is Cesàro summable

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(a) $\sum a_n$ converges $\implies \sum a_n$ is Cesàro summable and $\sum a_n = C[a_n]$
Fix $\varepsilon > 0$, let $\lim \sum a_n = \alpha$, and denote[2] the partial sums of $\sum a_n$ by (S_n). Because $\sum a_n$ is convergent, we know $\exists N_1 \in \mathbb{N}$ s.t $\forall n > N_1$ $d(S_n, \alpha) \leq \frac{\varepsilon}{2}$. By the Archemedian property (and the fact we know we can bound this particular finite sum), $\exists N_2$ s.t $\frac{|\sum_{i=0}^{N_1} S_i - \alpha|}{N_2 + 1} < \frac{\varepsilon}{2}$.

[2]credit to Greg for notation and sequencing of proof strategy. Stategy itself devised jointly. Our logic was that if you, for instance, take $1 + 2 + 3 + 3 + 3 + \ldots$ if you add enough 3s to the sum, the average of the series will be arbitrarily close to 3. We can apply this to any convergent series, where the partial sums are converging to something

Now for $n \in \mathbb{N}$ s.t $n \geq \max\{N_1, N_2\}$, by the triangle inequality

$$\begin{aligned}
\left|\frac{\sum_{i=0}^{N_1} S_i + \sum_{j=N_1+1}^{n} S_j}{n+1} - \alpha\right| &\leq \left|\frac{S_0 + \dots S_{N_1} - (1+N_1)\alpha}{n+1}\right| + \left|\frac{S_{N_1+1} + \dots + S_n + \alpha\big(n - (N_1+1)\big)}{n+1}\right| \\
&\leq \frac{|S_0 - \alpha| + \dots |S_{N_1} - \alpha|}{n+1} + \frac{|S_{N_1+1} - \alpha| + \dots + |S_n - \alpha|}{n+1} \\
&< \frac{\varepsilon}{2} + \frac{(n+1)\varepsilon}{2(n+1)} = \varepsilon
\end{aligned}$$

Now note that $C[a_n] = \lim \frac{\sum_{i=0}^{N_1} S_i + \sum_{j=N_1+1}^{N} S_j}{N+1}$. Therefore, the above logic shows that $C[a_n] \to \alpha$, meaning its convergent, and thus Cesàro summable, and also has the same value as $\sum a_n$

(b) $\sum a_n$ Cesàro summable $\implies \sum a_{n+1}$ is also, and $C[a_n] = a_0 + C[a_{n+1}]$
We will prove the second result ($C[a_n] = a_0 + C[a_{n+1}]$) which will imply the first.

Given $\sum a_n$ Cesàro summable. Let S_n' denote the partial sums of $\sum a_{n+1}$. Note that each S_n' starts at a_1 and ends with a_{n+1} we have

$$\begin{aligned}
C[a_n] = \lim \frac{S_0 + \dots + S_{N+1}}{N+1} &= \lim \frac{a_0 + (a_0 + a_1) + (a_0 + a_1 + a_2) + \dots (a_0 + \dots + a_{N+2})}{N+1} \\
&= \lim \frac{a_0 + a_0 + S_n' + \dots a_0 + S_{N+1}'}{N+1} \\
&= \lim \frac{(N+1)a_0}{N+1} + \lim \frac{\sum_{i=0}^{N+1} S_i'}{N+1} = a_0 + C[a_{N+1}]
\end{aligned}$$

If $\sum a_{n+1}$ was not Cesàro summable, then $C[a_{N+1}]$ would be a divergent term. Because a_0 is just some fixed sequence term, this would imply $C[a_n]$ was also divergent, meaning $\sum a_n$ would not be Cesàro summable, contradiction. Thus, $\sum a_n$ Cesàro summable $\implies \sum a_{n+1}$ is also.

(c) What is the Cesàro sum of $1 - 1 + 1 - 1 + \dots$
We have $S_0 = 1, S_1 = 0, S_2 = 1, S_3, = 0, \dots$. So for every even index, the partial sum is 1 and every odd index the partial sum is 0. So the sequence of $\frac{\sum_{i=0}^{N} S_i}{N+1}$ follows $1, \frac{1}{2}, \frac{2}{3}, \frac{2}{4} \dots$ for $N = 0, 1, 2, 3, \dots$. So all terms of the subsequence of odd indices are .5. The subsequence of even indices is decreasing and bounded by $[0, 1]$, so it must be convergent by the monotonic convergence theorem. If this subsequence converges to a limit that is not .5, that would be a contradiction because the subsequence of even and odd indices together make up the entire sequence, and limits must be unique. So the subsequence of even terms converges to .5, and the sequence does as well. Notice that $\lim \frac{\sum_{i=0}^{N} S_i}{N+1}$ is the Cesàro sum, so we have proved the Cesàro sum is .5.

(d) Using regularity, stability, and linearity, $1 + 2 + 4 + 8 + \dots$ is Cesàro summable $\implies$ its sum is -1
If $1 + 2 + 4 + 8 + \dots$ is Cesàro summable, then let its Cesàro sum be $C \in \mathbb{R}$. By part a, we also know that C is equal to the sum of the series. Thus, by the other properties we established (regualrity, stablility, and linearity), we can say

$$C = 1 + 2 + 4 + 8 + \dots = 1 + 2(1 + 2 + 4 + 8 + \dots) = 1 + 2C \implies C = -1$$

Clearly, the previous result establishes a contradiction. All the terms of the Cesaro sum are positive. So the limit of positive terms cannot possibly yield a negative number by the standard properties of real numbers. Therefore, it is not Cesàro summable. If you would like a more mathematical reason the the fact we arrived at a paradoxical result in a "proof by contradiction" fashion by assuming it was in fact summable, we can also see this by noting the sum of partial sums generalize to 2^N. Because for any M, n fixed $\exists m$ s.t $2^m > M2^n$, we know from an analogous example with factorials from class, $\exists N$ s.t $\forall n > N 2^n > Mn$. Because M can be arbitrarily large, this implies the Cesàro sum ($\lim \frac{2^{N+1}}{N+1}$) is diverging to ∞.

Weekly Homework 7

Paul B.
Math 531: Real Analysis I

April 23, 2022

1: Find the radius of convergence of the following:

Important Note: For these calculations, we will use $R = \limsup \left|\frac{a_n}{a_{n+1}}\right|$. This definition is in several other analysis texts and also implied by Rudin 3.37. Also, we will not rigorously prove these results, as we are only asked to calculate. Proofs of analogous results can be found in previous HWs.

(a) $\sum n^3 z^n$

$$R = \limsup \left| \left(\frac{n}{n+1}\right)^3 \right| = 1$$

(b) $\sum (.5^n n!)^{-1} z^n$

$$R = \limsup \left| \frac{.5^{n+1}(n+1)!}{.5^n n!} \right| = \limsup .5(n+1) = \infty$$

(c) $\sum (.5^n n^2)^{-1} z^n$

$$R = \limsup \left| \frac{.5^{n+1}(n+1)^2}{.5^n n^2} \right| = \limsup .5 \left(\frac{n+1}{n}\right)^2 = .5$$

(d) $\sum (3^n n^3)^{-1} z^n$

$$R = \limsup \left| \frac{3^{n+1}(n+1)^3}{3^n n^3} \right| = \limsup 3 \left(\frac{n+1}{n}\right)^3 = 3$$

2: Let $a, b, c \in \mathbb{R}^\infty$ s.t $c^k := a^k b^k$

(a) $p \in [1, \infty), a \in \ell^\infty, \text{and } b \in \ell^p \implies c \in \ell^p$
$a \in \ell^\infty$ means $\exists M \in \mathbb{R}^+$ s.t $\sup_k |a^k| < M$. $b \in \ell^p$ means $\sum |b^k|^p$ is finite. So[1]

$$\sum |c^k|^p = \sum |a^k b^k|^p \le \sum |M b^k|^p = M^p \sum |b^k|^p < \infty$$

implying that $c \in \ell^p$ by definition

[1] Please note for the rest of the assignment that, per the notation used in Rudin, $\sum x^k = \sum_{k=0}^{\infty} x^k$

(b) $a, b \in \ell^2 \implies c \in \ell^1$

For $a, b \in \mathbb{R}, |ab| = |a| \cdot |b|$. So by the Cauchy-Schwarz inequality

$$\left(\sum |c^k|\right)^2 = \left(\sum |a^k b^k|\right)^2 = \left(\sum |a^k| \cdot |b^k|\right)^2 \leq \sum (|a^k|)^2 \sum (|b^k|)^2 = ||a||_2^2 \cdot ||b||_2^2$$

Because the square root function preserves ordering

$$\implies ||c||_1 = \sum |c^k| = \sqrt{(\sum |c^k|)^2} \leq \sqrt{||a||_2^2 \cdot ||b||_2^2} = ||a||_2 \cdot ||b||_2 < \infty$$

since $a, b \in \ell^2 \implies ||a||_2, ||b||_2 < \infty$. Therefore, by definition $c \in \ell^1$

3:

$$e_n = (e_n^k)_{k \in \mathbb{N}}, e_n^k = \begin{cases} 1 & \text{if k=n} \\ 0 & \text{otherwise} \end{cases}$$

(a) For $p \in [1, \infty]$, show $(e_n)_{n \in \mathbb{N}}$ in ℓ^p is divergent

Given, $p \in [1, \infty]$, let $\varepsilon = \frac{||e^2 - e^1||_p}{2}$. Note that $\varepsilon > 0$ by construction. If our $p \in \mathbb{R}$ then $||e^2 - e^1||_p = 2^{\frac{1}{p}}$ because we will have two occurrences of 1^p in our infinite sum. If $p = \infty$, $||e^2 - e^1||_p = 1$ because the absolute difference between any two components of e_1 and e_2 will either be 1 or 0 (so sup across all k is 1). Further, note $\forall m, n \in \mathbb{N}$ s.t $m \neq n, ||e^2 - e^1||_p = ||e^m - e^n||_p$. This is because all will only have one component in each that is 1 and they will never be in the same place since $m \neq n$. So $\forall m, n$ s.t $m \neq n$

$$0 < \varepsilon < ||e_m - e_n||_p$$

So $(e_n)_{n \in \mathbb{N}}$ is not Cauchy, and is therefore divergent in ℓ^p, a complete space

(b) For k fixed, we will show $\lim_n e_n^k = 0$. Fix $k \in \mathbb{N}$. Now let $N > k$. Therefore, $\forall n \in \mathbb{N}$ s.t $n > N$, $e_n^k = 0$. So $\forall \varepsilon < 0$, $|e_n^k - 0| < \varepsilon$. So by definition, $\lim_n e_n^k = 0$

4: Show that $\{x \in \ell^1 | \sum x^k = 0\}$ is closed

Let $A = \{x \in \ell^1 | \sum x^k = 0\}$. We will show $A^c = \{x \in \ell^1 | \sum x^k \neq 0\}$ is open.

Given $x \in A^c \subseteq \ell^1$, we know $\sum x^k$ is absolutely convergent, thus convergent, so let $M = \sum x^n \neq 0$.

First, assume $M > 0$. Because open balls are defined with respect to ℓ^1, given $y \in B_{\frac{M}{2}}(x) \subseteq \ell^1$, by definition $\sum |x^k - y^k| < \frac{M}{2}$. Since $x^k - y^k \leq |x^k - y^k|$ and $y \in \ell^1$ (so $\sum y^k$ converges), we have

$$\sum x^k - \sum y^k = \sum x^k - y^k \leq \sum |x^k - y^k| < \frac{M}{2} \implies 0 < \frac{M}{2} = \sum x^k - \frac{M}{2} < \sum y^k$$

Thus, for $x \in A^c$, $\exists \varepsilon > 0$ s.t any $y \in B_\varepsilon(x)$ is in $\{x \in \ell^1 | \sum x^k = 0\}$, so $B_\varepsilon(x) \subseteq A^c$.

Now, assume $M < 0$ (consult above case for analogous details). Let $\varepsilon = |\frac{M}{2}|$ and $y \in B_\varepsilon(x)$. Then

$$\sum y^k - \sum x^k \leq \sum |y^k - x^k| < \varepsilon \implies \sum y^k < \sum x^k + \varepsilon = \frac{M}{2} < 0$$

Thus, for $x \in A^c$, $\exists \varepsilon > 0$ s.t any $y \in B_\varepsilon(x)$ is in $\{x \in \ell^1 | \sum x^k = 0\}$, so $B_\varepsilon(x) \subseteq A^c$.

So for all cases of $x \in A^c$, we can find an open ball that is a subset of A^c. So A^c is open.

5: Let $p \in [1, \infty]$. Show that any compact set in ℓ^p has empty interior

Proof by contradiction: given any $p \in [1, \infty]$ and $K \in \ell^p$ compact, suppose $K^\circ \neq \emptyset$. Then let $x \in K^\circ \subset \mathbb{R}^\infty$. Define open balls with respect to the p-norm. Then, by the definition of an interior, $\exists \varepsilon > 0$ s.t $B_\varepsilon(x) \subset K^\circ \subset K$. I proved on a previous homework $A \subset B \implies \overline{A} \subset \overline{B}$. We know $\overline{K} = K$ by the definition of compactness. So $\overline{B_\varepsilon}(x) \subset K$.

Because compactness implies sequential compactness, any infinite sequence in K has a subsequence that converges in K. We will show this is not the case for $\overline{B_\varepsilon}(x)$. Define[2] $y_n = x - \frac{\varepsilon}{2} e_n$. Then

$$||x - y||_p = ||\frac{\varepsilon}{2} e_n||_p = \varepsilon < 2$$

by linearity and $||e_n|| = 1$. So $y_n \in B_\varepsilon(x)$. Similar to #3, we know $\forall m, n \in \mathbb{N}$ s.t $m \neq n$

$$||y^m - y^n||_p = ||\frac{\varepsilon}{2}(e_n - e_m)||_p = \begin{cases} \frac{\varepsilon}{2} 2^{\frac{1}{p}} & \text{if } p \in \mathbb{R} \\ \frac{\varepsilon}{2} & \text{otherwise} \end{cases} \implies ||y^m - y^n||_p > \frac{\varepsilon}{4}$$

since $.5 < 2^{\frac{1}{p}}$ $\forall p \geq 1$. This holds $\forall$ non-equal m, n, so we cannot make a Cauchy (and therefore convergent) subsequence because *all* terms in *any* subsequence will be more than $\frac{\varepsilon}{4}$ apart. Thus, we have found an infinite sequence without a subsequence convergent in K, contradiction. So $K^\circ = \emptyset$.

6: Define $\ell_0^\infty = \{x \in \ell^\infty | \lim_k x^k = 0\}$. In ℓ^∞, $\overline{\mathbb{R}^\mathbb{N}} = \ell_0^\infty$

Just for simplified set notation, let $A = \ell_0^\infty$. Since we are in ℓ^∞, $A^c = \{x \in \ell^\infty | \lim_k x^k \neq 0\}$. The closure of $\mathbb{R}^\mathbb{N} \subseteq \ell^\infty$ is the set of all $x \in \ell^\infty$ s.t $\forall \varepsilon > 0$ $B_\varepsilon(x) \cap \mathbb{R}^\mathbb{N} \neq \emptyset$. We will show first $A \subset \overline{\mathbb{R}^\mathbb{N}}$ by $x \in A \implies x \in \overline{\mathbb{R}^\mathbb{N}}$. Then we will show $\overline{\mathbb{R}^\mathbb{N}} \subset A$ by $x \notin A \implies x \notin \overline{\mathbb{R}^\mathbb{N}}$ (contraposition).
Important note: Elements in $\mathbb{R}^\mathbb{N}$ have a finite number of non-zero terms. So given $y \in \mathbb{R}^\mathbb{N}$, for some N, $y^k = 0$ if $k > N$. Moreover, by defining balls with respect to the sup-norm since we're in ℓ^∞, if $y \in B_\varepsilon(x)$, then $\sup_n |x^n - y^n| < \varepsilon$, which means $\forall k$ $y^k \in (x^k - \varepsilon, x^k + \varepsilon)$

First, take $x \in A$ (so $\lim_k x^k = 0$).
Fix $\varepsilon > 0$. Then $\exists N$ s.t $\forall k > N$ $|x^k - 0| < \varepsilon$. Let $x' = (x^1, \ldots, x^N, 0, 0, \ldots)$. $x' \in B_\varepsilon(x)$ because $0 \in (x^k - \varepsilon, x^k + \varepsilon)$ $\forall k > N$. From above, $x' \in \mathbb{R}^\mathbb{N}$. So $x' \in B_\varepsilon(x) \cap \mathbb{R}^\mathbb{N}$, clearly meaning $B_\varepsilon(x) \cap \mathbb{R}^\mathbb{N} \neq \emptyset$. By the definition of a closure, $x \in \overline{\mathbb{R}^\mathbb{N}}$. So $A \subset \overline{\mathbb{R}^\mathbb{N}}$

Now, take $x \in A^c$ (so $\lim_k x^k \neq 0$)
Then by the definition of a limit and the Archedmedian property, for any $N' \in \mathbb{N}$ $\exists i > N'$ and $\varepsilon > 0$ s.t $|x^i| > \varepsilon$. We know, given $y \in \mathbb{R}^\mathbb{N}$, $\exists M$ s.t $\forall k > M$ $y^k = 0$. Thus, for some $i > M$ and $\varepsilon > 0$, $y_i = 0 \notin (x^i - \varepsilon, x^i + \varepsilon)$. So $y \notin B_\varepsilon(x)$ and, since this holds for any $y \in \mathbb{R}^\mathbb{N}$, $B_\varepsilon(x) \cap \mathbb{R}^\mathbb{N} = \emptyset$. By the definition of a closure, $x \notin \overline{\mathbb{R}^\mathbb{N}}$. Using contraposition ("not q implies not p"), $\overline{\mathbb{R}^\mathbb{N}} \subset A$

We have shown $A \subset \overline{\mathbb{R}^\mathbb{N}}$ and $\overline{\mathbb{R}^\mathbb{N}} \subset A$. Thus $\overline{\mathbb{R}^\mathbb{N}} = \ell_0^\infty$

[2]credit to Sarah for the construction of this sequence. Proof strategy was devised collectively

Weekly Homework 8

Paul B.
Math 531: Real Analysis I

April 23, 2022

2: For (X, d) metric space and $E \subset X$ closed and non-empty

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(a) $f : x \to R$ s.t $x \mapsto d(x, E)$ is continuous
Fix $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{2}, x \in X, z = B_\delta(x), \text{and}\, y \in Y$. By the triangle inequality

$$d(y, z) \leq d(x, y) + d(x, z) \;\; \underline{\textbf{and}} \;\; d(x, y) \leq d(x, z) + d(y, z)$$

Taking the inf over all $y \in E$ of each side of the inequalities

$$\implies d(z, E) \leq d(x, E) + d(x, z) \;\; \underline{\textbf{and}} \;\; d(x, E) \leq d(x, z) + d(z, E)$$

by definition of $d(\cdot, E)$. Since $z \in B_\delta(x)$ and $f(x) = d(x, E)$

$$\implies f(z) \leq f(x) + \frac{\varepsilon}{2} \;\; \underline{\textbf{and}} \;\; f(x) \leq \frac{\varepsilon}{2} + f(x) \implies f(z) - f(x), f(x) - f(z) \leq \frac{\varepsilon}{2}$$

Thus, $|f(x) - f(z)| < \varepsilon$ given $d(x, z) < \delta$. Since $d_{\mathbb{R}}(x, z) = |x - z|$, f is continuous.

(b) If $K \cap E = \emptyset$ and K compact, $d(X, E) > 0$
Proof by contradiction: assume $d(K, E) = 0$. Because K and E are disjoint, there must exist a sequence $(k_n) \in K$ s.t $d(k_n, E) \to 0$ (otherwise $\inf_{k \in K} d(k, E)$ would be positive). This will be the key for establishing the contradiction. Similarly, $\exists (e_n) \in E$ s.t $d(k_n, e_n) \to 0$.

By the completeness of $\mathbb{R}$, $\big(d(k_n, e_n)\big)$ is Cauchy. Fix $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t $\forall m, n > N$

$$d(k_n, e_n) < \frac{\varepsilon}{2} \;\; \underline{\textbf{and}} \;\; d(k_m, e_n) < \frac{\varepsilon}{2}$$

Then by the triangle inequality $d(k_n, k_m) < \varepsilon$. So (k_n) is Cauchy. Because compact sets are complete, $\exists k \in K$ s.t $k_n \to k$. By the continuity of f and Rudin 4.6, $f(k_n) \to f(k)$. Note from above that $f(k_n) \to 0$, and since limits are unique $f(k) = 0$. However, $E = \overline{E}$ since E is closed, so K and E disjoint $\implies K \cap \overline{E} = \emptyset$. $f(k) = 0$ iff $k \in \overline{E}$, but $k \in K$, contradiction.

3: For Y complete and continuous $f : E \to Y$ $(E \subset X)$,$\exists \overline{f} : \overline{E} \to Y$ s.t $\overline{f}(x) = f(x)\ \forall x \in E$

Let $x \in E'$. Then from #6, we know $\exists$ a sequence $(x_n) \in E$ s.t $x_n \to x$. So therefore, (x_n) is Cauchy, and for any $\sigma > 0 \exists M \in \mathbb{N}$ s.t $\forall m > n \geq M |x_m - x_n| < \sigma$ By this fact and uniform continuity, for each $\varepsilon > 0\ \exists \delta > 0$

$$|x_m - x_n| < \delta \implies |f(x_n) - f(x_y)| < \varepsilon$$

where we know this is a well-defined construction by letting $\sigma = \delta$ in our previous Cauchy result. Thus, we have also now shown $\big(f(x_n)\big)$ is also Cauchy. So $f(x_n) \to y$ for some $y \in \mathbb{R}$.

Therefore, define

$$\overline{f} : \overline{E} \to Y \text{ s.t } f(x) = \begin{cases} f(x) & x \in E \\ \lim\limits_{x_n \to x} f(x_n) & x \in E' \text{ (and } x_n \in E) \end{cases}$$

The work we've done shows this is well-defined. Now we need to show its continuous and unique.

Since f, any therefore $\overline{f}$, is continuous for $x \in E$, we just need to show the $x \in E'$ case. Let $(x_n), (z_n) \in E$ be sequences such that $x_n \to x, z_n \to z$. Note that $x, z \in E'$. So by construction, $f(x_n) \to \overline{f}(x)$ and $f(z_n) \to \overline{f}(z_n)$. Fix $\varepsilon > 0$. From previous HW, we know there some M satisfies both limit definitions: $\exists M \in \mathbb{N}$ s.t $\forall n > M\ d_Y(\overline{f}(x), f(x_n)), d_Y(\overline{f}(z), f(z_n)) < \frac{\varepsilon}{3}$. For $\sigma > 0$, impose $d_X(x, z) < \frac{\sigma}{2}$. We also know $d(x_n, z_n) \to d(x, z)$. So $\exists M' \in N$ s.t $\forall n > M'$ $|d_X(x_n, z_n) - d_X(x, z)| < \sigma$. Imposing $d_X(x, z) < \frac{\sigma}{2}$ and $n > M'$, $d_X(x_n, z_n) < \sigma$. By uniform continuity, $\exists \delta > 0$ s.t $d_X(x_n, z_n) < 2\delta \implies d_Y(f(x_n), f(z_n)) < \frac{\varepsilon}{3}$. Now let $N = \max\{M', M\}$. For $n > N$ and $d_X(x, z) < \delta$ (consider $\sigma = 2\delta$), by the triangle inequality (twice)

$$\begin{aligned} d_Y(\overline{f}(x), \overline{f}(z)) &\leq d_Y(\overline{f}(x), f(x_n)) + d_Y(\overline{f}(y), f(x_n)) \\ &\leq d_Y(\overline{f}(x), f(x_n)) + d_Y(\overline{f}(y), f(y_n)) + d_Y(f(x_n), f(z_n)) < \varepsilon \end{aligned}$$

where we used $d_X(x, z) < \delta \implies d_X(x_n, z_n) < 2\delta$. So we just showed that $n > N, x_n \to x$, $z_n \to z$, $d_X(x, z) < \delta \implies d_Y(\overline{f}(x), \overline{f}(z)) < \varepsilon$. We've now shown all $x \in \overline{E}$ cases, so $\overline{f}$ is continuous.

We now need to show uniqueness. We want to make sure there is not another extension, call it $\overline{g}$ that is not equal to $\overline{f}$. However, note that for $\overline{g}$ to be an extension of f, it must satisfy $\overline{g}(x) = f(x)\ \forall x \in E$. This implies $\overline{g}(x) = \overline{f}(x)\ \forall x \in E$. Trivially, E is dense in $\overline{E}$ (details in # 6). So by the second part of #6, $\overline{g}(x) = \overline{f}(x)\ \forall x \in E \implies \overline{g}(x) = \overline{f}(x)\ \forall x \in \overline{E}$. So $\overline{f}$ is unique.

4: Given $E = \{0, 2^0, 2^{-1}, 2^{-2}, \dots$ with the usual metric on $\mathbb{R}$

(a) For (X, d) metric space and $(a_n) \in X$, $\lim a_n = a$ iff the following is continuous: $f : E \to X$ s.t

$$f(x) :- \begin{cases} a_n & x = 2^{-n} \\ a & x = 0 \end{cases}$$

$\Longrightarrow$ Fix $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t $\forall n > N$ $d(a_n, a) > 0$. Let $\delta = 2^{-N} > 0$. Then $\forall n$ s.t $2^{-n} = |2^{-n} - 0| < \delta$, $d(f(x), 0)) = d(a_n, a) < \varepsilon$. Further, note that $\forall m$ s.t $m \neq n$ and $2^{-m} < \delta$, $|2^{-n} - 2^{-m}| < 2^{-n} < \delta$. Because (a_n) is convergent it's Cauchy, so by our previous construction, any such $|2^{-n} - 2^{-m}| < \delta \implies d(a_n, a_m) < \varepsilon$. Thus, f is continuous.

$\Longleftarrow$ Fix $\varepsilon > 0$. By continuity, $\exists \delta > 0$ s.t $|2^{-n}| < \delta \implies d(a_n, a) < \varepsilon$. $2^n > n$ (from class, $n \in \mathbb{N}$). So $n > \delta^{-1} \implies 2^{-n} < n^{-1} < \delta \implies d(a_n, a) < \varepsilon$. Thus, $\forall n > N = \delta^{-1}$, $d(a_n, a) < \varepsilon$, so $a_n \to a$.

(b) $f : X \to Y$ is continuous iff for every continuous $g : X \to Y$, $f \circ g : E \to Y$ is continuous
$\Longrightarrow$ Shown in class (Rudin 4.7)

$\Longleftarrow$ Each term in E is either 0 or takes the form 2^{-n} for some $n \in \mathbb{N}$. So given a sequence $(e_n) \in E$, we must have $e_n \to 0$. Because g is continuous, this implies by Rudin 4.6 that $g(e_n) \to g(0)$. Now let $a_n = g(e_n)$ and $a = g(0)$. We have shown that $a_n \to a$ in X. Further, because g can be *any* continuous function, we know that given $(a_n) \in X$, $\exists g : E \to X$ and $(e_n) \in E$ s.t $g(e_n) = a_n$. By $f \circ g$ continuous, since $a_n = g(e_n) \to g(0) = a$, $f(a_n) = f(g(e_n)) \to f(g(0)) = f(a)$. Because (a_n) can be any sequence in X, this shows that f is continuous

5: Given f on $\mathbb{R}$ s.t $\forall x$ $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$,$f$ is not necessarily continuous

Consider $f : \mathbb{R} \to \mathbb{R}$ s.t $f(x) = \begin{cases} x^{-2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. We know from class f is not continuous on $\mathbb{R}$ (the limit, from both the left and right, of $f(x)$ as $x \to 0$ does not exist). But at $x = 0$

$$\lim_{h \to 0}[f(x+h) - f(x-h)] = \lim_{h \to 0}[(h)^{-2} - (-h)^{-2}] = 0$$

6: Given $f, g : X \to Y$ continuous and $E \subset X$ dense

(a) $f(E)$ is dense in $f(X)$
Given $E \subset X$, from Rudin 2.18j, E is dense in X if each $x \in X$ is in E' or E. If $x \in E, f(x) \in f(E)$. So let $x \in E'$ and $U \subset f(x)$ some open neighborhood in Y centered around $f(x)$. By continuity, $f^{-1}(U)$ is open and contains x. Because $x \in E'$, $f^{-1}(U) \cap E \neq \emptyset$. So let $y \in f^{-1}(U) \cap E$. Then $f(y) \in U \cap f(E)$, so the intersection is non-empty, meaning $f(x)$ is a limit point of $f(E)$

If $x \notin X$, then $f(x) \notin f(X)$, meaning $z \in f(X) \implies z \in f(E) \cup f(E)'$. So $f(E)$ is dense in $f(X)$

(b) $g(p) = f(p)\ \forall p \in E \implies g(p) = f(p)\ \forall p \in X$
Given $p \in X$, from a) we know $p \in E \cup E'$. So first assume $p \in E$. Then we are already given $g(p) = f(p)$. Now assume $p \in E'$. Then by the definition of a limit point (Rudin 2.20 - any neighborhood contains infinitely many points of the set) and from class we know $\exists$ a sequence $(p_n) \in E$ that converges to p. Because $p_n \in E\ \forall n \in \mathbb{N}$, $g(p_n) = f(p_n)$. Further, by continuity, $\lim_{x\to p} f(x) = f(p)$ and $\lim_{x \to p} g(x) = g(p)$, so $g(p_n), f(p_n) \to f(p), g(p)$. However, because limits are unique and $g(p_n) = f(p_n) \forall n$, they must converge to the same point, implying $g(p) = f(p)$. We've now covered all possible $x \in X$, given $E \subset X$ dense.

7: Given f uniformly continuous function on $E \subseteq \mathbb{R}$ bounded

(a) f is bounded on E
Per approval from Dr. Stokols, we will just use our powerful result from #3 to make this follow pretty quickly. First, note that the closure of E is closed. It is also bounded because its the smallest closed set containing E, and we know from class for any bounded set $E\ \exists R \in R^+$ s.t $E \subsetneq [-R, R]$. So therefore, $\overline{E}$ is compact. From #3, because f is uniformly continuous, the extension $\overline{f}$ is continuous on $\overline{E}$. By a theorem from class, because $\overline{f}$ is continuous and $\overline{E}$ is compact, $\overline{f}(\overline{E})$ is compact. From #3: each $y \in f(E)$ is in $\overline{f}(\overline{E})$, so $f(E) \subset \overline{f}(\overline{E})$, meaning it's bounded.

(b) f is not necessarily bounded if E is unbounded
Let $E = \mathbb{Q}^+$. Consider[1] $f(x) = 3x + 7$. Fix $\varepsilon > 0$. Then for $\delta = \varepsilon/3$ and $x, y \in E$

$$|x - y| < \delta \implies |f(x) - f(y)| = 3|x - y| < 3\delta = \varepsilon$$

So f is uniformly continuous. However, f is unbounded. Pick any $M \in R^+$. Then we know $\exists q \in \mathbb{Q}^+$ s.t $q > M$. Further, $M < q < f(q)$ by construction ($x \geq 0 \implies x < f(x)$).

8: Let $I = [0, 1]$. $f : I \to I$ continuous $\implies f(x) = x$ for some $x \in I$

First note the trivial cases: if $f(1) = 1$ or $f(0) = 0$ we are done. Assume neither of those are the case. Define $g : I \to [-1, 1]$ s.t $g(x) = f(x) - x$. g is continuous because it's the sum of continuous functions (f is continuous and we know the identity map is continuous from class). Because $f(x) \geq 0$ and $f(0) \neq 0$, $g(0) > 0$. Further, because $f(x) \leq 1$ and $f(1) \neq 1$, $g(1) < 0$. So $g(1) < 0 < g(0)$. Let $c = 0 \in (g(1), g(0))$. Then by IVT $\exists x \in (0, 1)$ s.t $g(x) = c = 0$. Therefore by construction, we have proved $\exists x \in I$ s.t $f(x) = x$

[1] https://people.math.wisc.edu/~robbin/521dir/cont.pdf

Weekly Homework 9

Paul B.
Math 531: Real Analysis I

April 23, 2022

1: There exists $f : [0, 1) \to \mathbb{R}$ continuous and surjective

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Because our function has to include 0, instead of thinking about a piecewise function, we want some kind of oscillation that captures all the reals. So we make use of the topologist's sin curve[1] and note

$$f(x) = \frac{1}{1-x} \sin\left(\frac{1}{1-x}\right)$$

is both continuous on $[0, 1)$ and has the property $f([0, 1)) = \mathbb{R}$. Continuity is clear, but to offer some justification for surjectivity, note that we cannot bound f from above or below (and by continuity+IVT, captures everything else when oscillating across the x-axis) so it captures all of $\mathbb{R}$.

2: With X metric space, $f : \mathbb{R} \to X$ is continuous on $\mathbb{R}^*$ if $\lim_{x \to \pm\infty} f(x) \in X$

Let f be continuous on $\mathbb{R}, \mathbb{R}^*$ and define $\alpha = \lim_{x\to\infty} f(x)$ and $\beta = \lim_{x\to\infty} f(x)$

(a) If $X = \mathbb{R}$, f is bounded
By the definition of a limit, for each $\varepsilon > 0$ $\exists N \in \mathbb{R}^-$ s.t $\forall x < N$ $|f(x) - \beta| < \varepsilon$. Therefore $f(x) < \varepsilon + \beta \implies |f(x)| < |\beta| + \varepsilon$. So let $\varepsilon = 1$ and define $R_1 < 0$ s.t $x < R_1 \implies |f(x)| < |\beta| + 1$. We can similarly define $R_2 > 0$ s.t $x > R_2 \implies |f(x)| < |\alpha| + 1$. We have now shown that if $x \in (-\infty, R_1) \cup (R_2, \infty)$, then $f(x)$ is bounded. The only cases of $\mathbb{R}$ we have not accounted for are $x \in [R_1, R_2]$. However, we know continuous functions map compact sets to compact sets, and since $X = \mathbb{R}$, combining this information means $f([R_1, R_2])$ is compact and in $\mathbb{R}$, therefore bounded, so $\exists B \in \mathbb{R}^+$ s.t $x \in [R_1, R_2] \implies f(x) < B$. Take $M = \max\{|\alpha| + 1, |\beta| + 1, B\}$. We have now shown all cases: given $x \in \mathbb{R}$, $|f(x)| < M$. Therefore, $f(\mathbb{R})$ is bounded.

(b) In general, $f(\mathbb{R}) \cup \{\alpha, \beta\}$ is compact in X
Let $K = f(\mathbb{R}) \cup \{\alpha, \beta\}$ and note $K \subseteq X$. This proof is analogous to the proof in HW #3 problem 4: constructing a finite subcover for any arbitrary open cover of $\{\frac{1}{n} | n \in \mathbb{N}\} \cup \{0\}$.

Let $\{G_\sigma\}_{\sigma \in A}$ be an open cover of K. Since $\alpha \in K, \exists a \in A$ s.t $\alpha \in G_a$. By G_a open, $\exists r > 0$ s.t $B_r(\alpha) \subseteq G_a$. Also, since $f(x) \overset{\infty}{\to} \alpha$, given $\varepsilon > 0$, $\exists M > 0$ s.t $\forall M' > M$ $d_X(\alpha, f(M')) < \varepsilon$. So if $x > M$, $f(x) \in B_r(\alpha) \subseteq G_a$. We can repeat a similar set of steps for β to show that for $b \in A$ s.t $\beta \in G_b$, there exists $N < 0$ s.t if $x < N$, $f(x) \in G_b$.

[1] Credit to Sarah for this example. Also, Dr. Stokols clarified a rigorous proof wasn't needed

Continuous functions map compact sets to compact sets. so $f([N,M])$ compact. Because $f([N,M]) \subset K$, $\{G_\sigma\}_{\sigma\in A}$ is an open cover for $f([N,M])$, and by compactness there exists a finite set $B \subset A$ s.t $\{G_\sigma\}_{\sigma\in B}$ covers $f([N,M])$. Recall from above: if $x \in \mathbb{R}$ but $x \notin [N,M]$, we know $f(x) \in G_a \cup G_b$. Note $f(\mathbb{R}) = \{f(x)|x \in [N,M]\} \cup \{f(x)|x \in \mathbb{R}\backslash[N,M]\}$, so we have done the work to create a finite subcover for $f(\mathbb{R})$ and are done. Explicitly: given any arbitrary open cover for K ($\{G_\alpha\}_{\alpha\in A}$), we can find a a finite set $C = \{a,b\} \cup B$ that yields $\{G_\sigma\}_{\sigma\in C}$, a finite subcover.

3: If $|f(x) - f(y)| \le (x-y)^2 \ \forall x, y$, then f is constant

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Note that $(x-y)^2 = |x-y|^2$. Therefore, dividing through by $|x-y|$ yields

$$|f(x)-f(y)| \le (x-y)^2 \implies 0 \le \left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y| \implies 0 \le \lim_{x\to y}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{x\to y}|x-y| = 0$$

By the squeeze theorem, $|f'(x)| = \lim_{x\to y}\left|\frac{f(x)-f(y)}{x-y}\right| = 0$. So f is differentiable for all $x \in \mathbb{R}$, meaning we can apply Rudin 5.11 and conclude f is constant on $\mathbb{R}$

4: Suppose $f'(x) > 0$ on (a,b)

(a) f is strictly increasing on (a,b)
For $d, e \in (a,b)$ (WLOG $e > d$) f is differentiable (so continuous) on $[d,e]$. By MVT $\exists c \in (d,e)$ s.t

$$f(e) - f(d) = f'(c)\cdot(e-d) > 0$$

since $f'(c), e-d > 0$. Thus for $d, e \in (a,b)$, $f(e) > f(d)$ if $e > d$, so f strictly increasing on (a,b)

(b) For $g(\cdot) = f^{-1}(\cdot)$, g is differentiable and $g'(f(x)) = (f'(x))^{-1}$ $(x \in (a,b))$
Note that part a shows f is 1-1 because for any $d, e \in (a,b)$ s.t $d \ne e$, $f(d) \ne f(e)$, meaning g, the inverse function of f, is well-defined (implied by Rudin 4.17).

Fix $x \in (a,b)$ and define $y = f(x)$ and $z \ne y$ s.t $z \in (\lim_{\alpha\to a^+} f(\alpha), \lim_{\beta\to b^-} f(\beta))$. By f continuous on (a,b), define $t \in (a,b)$ s.t $f(t) = z$. So $f(t) \ne f(x)$ $(z \ne y)$. By the continuity of f (Rudin 4.6), $t \to x \implies z = f(t) \to f(x) = y$. Therefore, by construction and definition of the inverse

$$\frac{g(f(x)) - g(f(t))}{f(x) - f(t)} = \left(\frac{f(x)-f(t)}{x-t}\right)^{-1}$$

$$\implies 0 < (f'(x))^{-1} = \lim_{t\to x}\left(\frac{f(x)-f(t)}{x-t}\right)^{-1} = \lim_{z\to y}\frac{g(f(x) - g(f(t))}{f(x)-f(t)} = g'(y) = g'(f(x))$$

Because the first term on the bottom expression is defined, this means $g'(f(x))$ is defined for all $x \in (a,b)$, meaning g is differentiable and $g'(f(x)) = (f'(x))^{-1}$.

5: For g with bounded derivative, $f(x) = x + \varepsilon g(x)$ is 1:1 if ε is small enough

From the form of f and the differentiability of g, by Rudin 5.3 we have

$$f'(x) = \lim_{h\to 0} \frac{x+h-x}{h} - \varepsilon g'(x) = 1 - \varepsilon g'(x)$$

By boundedness, $\exists M > 0$ s.t $|g'(x)| < M$, which means for $\varepsilon \in (0, \frac{1}{M})$

$$f'(x) = 1 - \varepsilon g'(x) \geq 1 - \varepsilon M > 0$$

$f'(x) > 0 \implies f$ is 1:1 by the MVT because we know for any $a, b \in \mathbb{R}^+$ we know $\exists c \in (a, b)$ s.t

$$f(b) - f(a) = f'(c) \cdot (b - a) \neq 0$$

showing $f(a) \neq f(b)$ for any $a \neq b$. So f is 1:1 if $\varepsilon > 0$ is small enough ($< M^{-1}$).

6: $f : \mathbb{R}^+ \to \mathbb{R}$ s.t $f'(x) \overset{\infty}{\to} 0$. Then $g(x) = f(x+1) - f(x) \overset{\infty}{\to} 0$

Fix $\varepsilon > 0$. By $f'(x) \overset{\infty}{\to} 0$, $\exists M > 0$ s.t $x > M \implies |f'(x)| < \varepsilon$. Also, by MVT, $\exists c \in (x, x+1)$ s.t

$$f(x+1) - f(x) = f'(c) \cdot (x + 1 - x) = f'(c)$$

If we take, $x > M$, then $c > M$ so

$$|g(x) - 0| = |f(x+1) - f(x)| = |f'(c)| < \varepsilon$$

By the definition of the limit, this implies $g(x) \overset{\infty}{\to} 0$

7: If $f'(x)\&g'(x)$ exists, $g'(x) \neq 0,$ and $f(x) = g(x) = 0$ then $f'(x)/g'(x) = \lim_{t\to x} f(t)/g(t)$

The relevant derivatives exist and $g'(x) \neq 0$ so

$$\frac{f'(x)}{g'(x)} = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \cdot \frac{t-x}{g(t)-g(x)} = \lim_{t\to x} \frac{f(t)-f(x)}{g(t)-g(x)} = \lim_{t\to x} \frac{f(t))}{g(t)}$$

since $g(x) = f(x) = 0$ and using the product rule of limits.

Weekly Homework 10

Paul B.
Math 531: Real Analysis I

April 23, 2022

1: Suppose f is defined in some $B_r(x)$ $(r > 0)$ and $f''(x)$ exists

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(a) Prove $f''(x) = \lim_{h\to 0} h^{-2}\big(f(x+h) + f(x-h) - 2f(x)\big)$
Let $\phi(h) = f(x+h) + f(x-h) - 2f(x)$ and $g(h) = h^2$, where $\phi(\cdot)$ is only (necessarily) defined if $x \pm h$ is in the neighborhood of x s.t f is defined. Then

$$\lim_{h\to 0}\frac{\phi'(h)}{g'(h)} = \lim_{h\to 0}\frac{f'(x+h) - f'(x-h)}{2h} = \lim_{h\to 0}\left(\frac{f'(x+h) - f'(x)}{2h} + \frac{-f'(x-h) + f'(x)}{2h}\right) = f''(x)$$

since $f''(x)$ exists and we can pull out a constant .5 to match its exact definition[1]. Note that $\phi(h), g(h) \overset{0}{\to} 0$. So by Rudin 5.13 $\lim\limits_{h\to 0}\frac{\phi(h)}{g(h)} = \lim\limits_{h\to 0}\frac{\phi'(h)}{g'(h)} = f''(x)$.

(b) Prove the above limit may exist even if $f''(x)$ doesn't
Consider $f(x) = 5$ if $x > 0$, 0 if $x = 0$, and -5 if $x < 0$. This function is clearly not continuous at 0, so its second derivative at 0 cannot exist. However, the limit is clearly 0 as $h \to 0$. This is because for any $h \neq 0$, $f(x+h) - f(x-h) = 0 = f(x)$. So $\forall h \neq 0$, $\phi(h)/h^2 = 0$ (using notation from the first part). Considering a formal limit as $h \to 0$ definition (i.e. $|h| < \varepsilon$), the limit exists and is 0.

2: Let $f : (a, b) \to \mathbb{R}$

(a) f is convex iff f' monotonic increasing
We will do an iff through the whole proof so one can start at the beginning for assuming convexity or start at the end for assuming increasing (and it will follow both ways). In order to do this a bit more cleanly, we will go ahead and get a MVT result. For $\lambda \in (0, 1)$, let $c = (1-\lambda)a + \lambda b$. By the MVT, $\exists x \in (a, c)$ and $y \in (c, b)$ s.t

$$f'(x) = \frac{f(c) - f(a)}{c - a} \ \underline{\textbf{and}}\ f'(y) = \frac{f(b) - f(c)}{b - c}$$

We will refer to these definitions later in proof.

[1] note that the two terms correspond to limits "from the left" and "from the right" at x, which must be equivalent since the exists derivative, and therefore this particular limit, exists. This was more or less established in class as well.

Now note that, in general,

$$f(c) \leq (1-\lambda)f(a)+\lambda f(b) \iff f(c) \leq \frac{c-b}{a-b}f(a)-\left(1-\frac{c-b}{a-b}\right)f(b) \iff \frac{f(c)-f(a)}{c-a} \leq \frac{f(b)-f(c)}{b-c}$$

since $\lambda = \frac{c-b}{a-b}$ by construction. Now note, from our previous definitions

$$\frac{f(c)-f(a)}{c-a} \leq \frac{f(b)-f(c)}{b-c} \iff f'(x) \leq f'(y)$$

and by construction $x < y$. Now look what we've proven: if you start with $f(c) \leq (1-\lambda)f(a) + \lambda f(b)$ you get $x < y \implies f'(x) \leq f'(y)$, and vice versa.These are respectively the definitions of convexity and monotonic increasing[2], so our proof works starting from the beginning and working forward or starting at the end at working back.

(b) If $f''(x)$ exists $\forall x \in (a, b)$, f is convex iff $f''(x) \geq 0$ $(x \in (a, b))$
We proved on the previous HW that f is increasing iff $f'(x) \geq 0$. So by extension, from part a

$$f \text{ is convex} \iff f' \text{ is increasing} \iff f''(x) \geq 0 \;\; \forall x \in (a, b)$$

3: f twice differentiable on (a, ∞) with M_0, M_1, M_2 the respective sup of $|f(x)|, |f'(x)|, |f''(x)|$

(a) $M_1^2 \leq 4M_0M_2$
First, note that if $M_0 = 0$, then $\nexists x \in (a, \infty)$ s.t $f(x) \neq 0$. Therefore, $f = 0$ on (a, ∞), meaning $M_1 = 0$ so the inequality holds. Now assume $M_0 \neq 0$.

Per the hint given in Rudin, by Taylor's Theorem, given $h > 0$, $|f'(x)| \leq hM_2 + M_0h^{-1}$ for any $x \in (a, \infty)$. Since this holds for all x, it is an upper bound, and therefore must be greater than or equal to the least upper bound, yielding $M_1 \leq hM_2 + M_0h^{-1} \implies M_1 \leq h^2M_2 + 2M_2M_0 + M_0h^{-2}$ by squaring both sides. Basic algebra shows $\alpha^2x^2 + \beta^2x^{-2} = 2\alpha\beta$ has the solution $x = \pm\sqrt{\alpha/\beta}$. So with respect to our problem, letting $h = \sqrt{M_2/M_0}$ leads to $h^2M_2 + M_0h^{-2} = 2M_2M_0$. This is a valid definition for $h > 0$ as long as $M_0 \neq 0$, which we have assumed. Therefore selecting this h, $M_1 \leq h^2M_2 + 2M_2M_0 + M_0h^{-2} = 4M_0M_2$.
$M_0 \geq 0$ by construction, so we've shown all cases and the inequality holds.

(b) $M_0 = 1, M_1 = 4, M_2 = 4$ for $f(x) = 2x^2 - 1$ for $x \in (-1, 0)$ and $(x^2 - 1)(x^2 + 1)^{-1}$ for $x \in [0, \infty)$
We will look at each section of the piecewise individually and combine what we know at the end.

If $x \in (-1, 0)$: $\sup|f(x)| = 1$ because if $a \in (0, 1)$, for $x \in \left(-1, -\sqrt{\frac{a+1}{2}}\right)$ $2x^2 + 1 > a$. From class $f'(x) = 4x$, so $\sup|f'(x)| = 4$ $(a \in (0, 4),\, x \in (-1, -\frac{a}{4}) \implies |4x| > a)$. Trivially, $\sup|f''(x)| = 4$

If $x \in [0, \infty)$: $\sup|f(x)| = 1$ $\left(a \in (0, 1), x \in \left(\sqrt{\frac{2}{1-a}}, \infty\right) \implies f(x) > a\right)$. By Rudin 5.3, $f'(x) = 4x(x^2 + 1)^{-2}$. On $[0, 1]$, $|f'(x)| \leq \frac{4\cdot 1}{(0^2+1)^2} = 2$. Otherwise, $a \in (0, 1), x \in (\frac{2+\sqrt{4-a^2}}{a}, \infty)$ gives $|f'(x)| > a$. So $\sup|f'(x)| \leq 2$. Again by Rudin 5.3, $f''(x) = \frac{4(1-3x^2)}{(x^2+1)^3}$. If $x \in [0, 1)$, then $|f''(x)| \leq \frac{4(1-3\cdot 0^2)}{(0^2+1)^3)} = 4$. Otherwise $(x \geq 1)$, by Rudin 5.13 $f''(x) = \frac{4}{(x^2+1)^3} + \frac{-12x^2}{(x^2+1)^3} \to 0$ and $-2 = f''(1) \leq f''(x) < 0$, so $\sup|f''(x)| \leq 4$

The sup agrees for $|f(x)|, |f''(x)|$ and the first sup of $|f'(x)|$ dominates, so $M_0 = M_2 = 4, M_1 = 1$.

[2]This holds for all arbitrary c and consequently $x, y \in [a, b]$ (we can also restrict the interval)

4: $f : [a, b] \to \mathbb{R}$, $n-1$ times diff. Let α, β from 5.15 and $Q(t) = \frac{f(t)-f(\beta)}{t-\beta}$. For $t \in [a, b], t \neq \beta$ differentiate $f(t) - f(\beta) = (t-\beta)Q(t)$ n-1 times at $t = \alpha$ and show $f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n$

By the differentiability of f, Q(t) is differentiable at $\alpha \neq \beta$ by Rudin 5.3. So differentiating both sides of $f(\alpha) - f(\beta) = (\alpha - \beta)Q(\alpha)$

$$f'(\alpha) = -(\beta - \alpha)Q'(t) + Q(t)$$

Differentiating further yields

$$f''(\alpha) = -(\beta - \alpha)Q''(t) + 2Q'(t) \implies \cdots \implies f^{(k)}(\alpha) = -(\beta - \alpha)Q^{(k)}(\alpha) + kQ^{(k-1)}(\alpha)$$

for $k \leq n-1$. Multiplying by $\frac{(\beta-\alpha)^k}{k!}$

$$\frac{(\beta - \alpha)^k}{k!} = -\frac{(\beta - \alpha)^{k+1}Q^{(k)}(\alpha)}{k!} + \frac{(\beta - \alpha)^k Q^{(k-1)}(\alpha)}{(k-1)!}$$

The right hand side partial sums has a pattern: for $S_n = \sum_{k=1}^{n} \frac{(\beta-\alpha)^k}{(k-1)!}\left(Q^{(k-1)}(\alpha) - \frac{(\beta-\alpha)Q^{(k)}(\alpha)}{k}\right)$

$$S_2 = \frac{(\beta - \alpha)^1 Q^{(1)}(\alpha)}{1!} - \frac{(\beta - \alpha)^3 Q^{(2)}(\alpha)}{2!} \implies \cdots \implies S_{n-1} = -\left(f(\alpha) - f(\beta) + \frac{(\beta - \alpha)^n Q^{(n)}(\alpha)}{n!}\right)$$

by the definition of $Q(\alpha)$. So taking $\sum_{k=1}^{n-1}$ of both sides of the original equality leads to

$$\sum_{k=0}^{(n-1)} \frac{(\beta - \alpha)^k f^{(k)(\alpha)}}{k!} = \psi(n-1) \implies f(\beta) = \sum_{k=0}^{(n-1)} \frac{(\beta - \alpha)^k f^{(k)(\alpha)}}{k!} + \frac{(\beta - \alpha)^n Q^{(n)}(\alpha)}{n!}$$

where we moved $f(\alpha)$ to the left hand side from $\psi(n-1)$ to complete the definition of $P(\beta)$.

5: f diff on $[a, b], f(a) = 0, \exists A \in \mathbb{R}$ s.t $|f'(x)| \leq A|f(x)|$. $\forall x \in [a, b]$, $f(x) = 0$

Let $M_0 = \sup |f(x)|, M_1 = \sup |f'(x)|$. First note that from work we did in #3a we know that $|f'(x)| \leq A|f(x)\ \forall x \in [a, b] \implies M_1 \leq AM_0$. We will first rigorously prove the hint given by Rudin and then show the implication when combined with the continuity of f (implied by differentiability).

For some $r > 0$, $Ar = 1$. So fix $x_0 \in (a, b]$ s.t $x_0 < r + a$, so $A(x_0 - a) < 1$. For $x \in [a, x_0]$, by the MVT $\exists c \in (a, x)$ s.t $|f(x)| = |f(x) - f(a)| \leq |f'(c)(x - a)| \leq |f'(c)|(x_0 - a)$. Since $|f'(c)| \leq M_1$ ($\forall c \in [a, b]$) and $M_1 \leq AM_0$, we have $|f(x)| \leq AM_0(x_0 - a)$. Equivalently, by our assumption, for some $\alpha \in (0, 1)$ $|f(x)| \leq \alpha M_0 < M_0\ \forall x \in [a, x_0]$. Because M_0 is a least upper bound, this implies that $|f(x)|$ (and therefore $f(x)$) is exactly 0 on $[a, x_0]$ (otherwise we would arrive at an immediate contradiction: αM_0 would be a smaller upper bound).

Per our previous definition, now let $z = r + a$. For a contradiction, assume $f(z) \neq 0$. The implication of the work we've done is that f is exactly 0 on $[a, z)$ (if we pick any $x_0 \in (a, z), f = 0$ on $[a, x_0]$). Fix $\varepsilon = .5|f(z)|$. By the continuity of f, for $x_0 \in (a, z)$ $\exists \delta > 0$ s.t

$$|z - x_0| < \delta \implies |f(z) - f(x_0)| = |f(z)| < \varepsilon = .5|f(z)|$$

To be clear[3]: this violation of continuity occurs because *any* $x_0 \in (a, z)$, no matter how small $|z - x_0|$ is, will have the property $f(x_0) = 0$. So we have our contradiction. We have now expanded the interval where $f = 0$ and can keep applying the same argument to show that if we assume that $\exists h > 0$ s.t $f(z + h) = 0$, we will arrive at a contradiction. Thus, $f = 0$.

6: $\phi : [a, b] \times [\alpha, \beta] \to \mathbb{R}$. A *solution* is f diff on $[a, b]$ s.t $f(a) = y(a) = c \in [\alpha, \beta]$, $f(x) \in [\alpha, \beta]$, and $f'(x) = \phi(x, f(x))$. Max one solution if $\exists A \in \mathbb{R}$ s.t for $(x, y_1), (x, y_2) \in \mathbb{R}$, $|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$

We WTS that given this condition, for any solutions f and g, $f = g$. So let f and g be two solutions and , per the hint, define $d(x)$ as the difference between f and g at x. We will show $d(x) = 0 \ \forall x$.

It is also important to be careful about the notation here. What we have is an ordinary differential equation problem. So as an abstraction, consider the example given in the hint of $\sqrt{y}$. We have ODE(y)=$y' - \sqrt{y}$. We are looking for a solution f s.t ODE($f(x)$)=0. So substituting $y' = \phi(x, f(x))$ and noting d is differentiable by the differentiability of solutions and Rudin 5.3

$$|d'(x)| = |f'(x) - g'(x)| = |\phi(x, f(x)) - \phi(x, g(x))| \leq A|f'(x) - g'(x)| = A|d(x)|$$

where we used substitution and the inequality given in the problem. We are also given that $f(a) = g(a) = c$ since they're solutions, so $d(a) = 0$. Thus by #5, $d(x) = 0 \ \forall x \in [a, b]$

[3] This is similar to what we saw in #1 b

Weekly Homework 11

Paul B.
Math 531: Real Analysis I

April 28, 2022

1: (Bounded Variation - BV properties)

(a) Any Lipschitz continuous function is BV
Let[1] $f : [a,b] \to \mathbb{R}$ Lipschitz continuous. Then $\exists M \in \mathbb{R}$ s.t $|f(x) - f(y)| \leq M|x - y|$ $(\forall x, y \in X)$.
Let $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ be a generic partition of $[a,b]$. Then by definition

$$TV(f) = \sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \sup_P \sum_{i=1}^{n} M|x_i - x_{i-1}| = M \cdot TV(id(x)) = M(b-a)$$

By the monotonicity of $id(x) = x$. $TV(f) \in [0, M(b-a)]$, so f Lipschitz continuous are BV.

(b) Any BV function is bounded.
Given $f : [a,b] \to \mathbb{R}$ and P the aforementioned generic partition, for $x \in [a,b]$ and some $M \in \mathbb{R}$

$$|f(x) - f(a)| \leq \sup_P \sum_{i=1}^{n} |f(x_i) - f(x_[i-1])| < M$$

since we are taking the supremum over all possible partitions, so we could consider any x in such a partition to have a guaranteed upper bound. Therefore by the triangle inequality

$$|f(x)| = |f(x) - f(a) + f(a)| \leq |f(a)| + |f(x) - f(a)| \leq |f(a)| + \sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

Because $f(a) \in \mathbb{R}$ by assumption, $|f(x)| < M + |f(a)| < \infty$ $(\forall x)$, so f is bounded.

(c) For $d(f,g) = TV(f - g)$, $BV([a,b], d)$, set of all BV functions $[a,b] \to \mathbb{R}$, isn't a metric space
Take $f(x) = c$ for $c \in \mathbb{R} \backslash 0$ and $g(x) = 0$ $(\forall x)$. They have "no variation" $(f, g \in BV([a,b], d))$. But

$$d(f,g) = TV(f - g) = TV(c) = 0$$

Metrics must follow $d(f,g) = 0 \iff f = g$. However, $f \neq g$, so $BV([a,b], d)$ isn't a metric space

[1] Credit to Marc for this proof strategy

2: Let $f, \alpha : [a,b] \to \mathbb{R}$ with $\alpha = a^+ - \alpha^-$ s.t α^+, α^- monotone increasing (so α is BV).
Recall that $\mathcal{R}(\alpha) = \mathcal{R}(\alpha^+) \cap \mathcal{R}(\alpha^-)$ and $\int f d\alpha = \int f d\alpha^+ - \int f d\alpha^-$.

(a) If $f \in \mathcal{R}(\alpha)$, $|\int f d\alpha| \leq [\sup_{x \in [a,b]} |f(x)|] \cdot TV(\alpha)$
By $f \in \mathcal{R}(\alpha)$ and α BV, $\int f d\alpha \leq U(f, P, \alpha)$ (any P). Therefore

$$\left|\int f d\alpha\right| \leq |U(f,P,\alpha)| = \left|\sum_{i=1}^n M_i \Delta\alpha_i\right| \leq [\sup_{x \in [a,b]} |f(x)|] \sum_{i=1}^n |\alpha_i - \alpha_{i-1}| \leq [\sup_{x \in [a,b]} |f(x)|] \cdot TV(\alpha)$$

since $B \subseteq A \implies \sup_B f(x) \leq \sup_A f(x)$.

(b) Define $\overline{\alpha} = \alpha^+ + \alpha^-$ increasing. $\mathcal{R}(\alpha) = \mathcal{R}(\overline{\alpha})$ and for $f \geq 0$ $|\int f d\alpha| \leq \int f d\overline{\alpha}$
We know that $\mathcal{R}(\alpha) = \mathcal{R}(\alpha^+) \cap \mathcal{R}(\alpha^-)$ and $\int f d\alpha = \int f d\alpha^+ - \int f d\alpha^-$. Trivially note that if $f \in \mathcal{R}(\alpha^+) \cap \mathcal{R}(\alpha^-)$, then $f \in \mathcal{R}(\alpha^+), \mathcal{R}(\alpha^-)$. Therefore by Rudin 6.12e $f \in \mathcal{R}(\alpha^+ + \alpha^-) = \mathcal{R}(\overline{\alpha})$. If we instead assume $f \in \mathcal{R}(\alpha^+ + \alpha^-)$, then $\int f d\overline{\alpha} = \int f d\alpha^+ + \int f d\alpha^-$. This implies that both of those objects exist, so $f \in \mathcal{R}(\alpha^-), \mathcal{R}(\alpha^+)$ thus $f \in \mathcal{R}(\alpha)$. Combining cases $\mathcal{R}(\alpha) = \mathcal{R}(\overline{\alpha})$
Finally, by 6.12

$$\int f d(\overline{\alpha}) = \int f d(\alpha^+) + \int f d(\alpha^-)$$

$$\implies \left|\int f d\alpha\right| = \left|\int f d\alpha^+ - \int f d\alpha\right| \leq \left|\int f d\alpha^+\right| - \left|\int f d\alpha\right|$$

using the above definitions, the triangle inequality, and $f \geq 0$ (w/ α^+, α^- increasing).

3: A set $E \subseteq \mathbb{R}$ is measure 0 if for each $\varepsilon > 0 \exists$ a finite set of open balls s.t $E \subseteq \cup_{i=1}^n B_{\delta_i}(p_i)$ and $\sum_{i=1}^n \delta_i < \varepsilon$. $f : [a,b] \to \mathbb{R}$ is continuous almost everywhere if its set of discontinuities is measure 0

(a) Any bounded function that is continuous almost everywhere is in $\mathcal{R}$
We follow the proof of 6.10 (similar result) but use more notation to make some steps more rigorous.

Fix $\varepsilon > 0$. Let $\sigma = \frac{\varepsilon}{2(b-a)}$, a valid construction since a, b are fixed. Because f is bounded, note that for $L = \sup |f(x)| \in \mathbb{R}^+$ and[2] any $[x_{i-1}, x_i] \subset [a,b]$, $M_i - m_i < 2L$. So let $\varepsilon' = \min\{\frac{\varepsilon}{2}, \frac{\varepsilon}{4L}\}$.

By the measure 0 property, there exists $\{\delta_i\}_{i=1}^n, \{p_i\}_{i=1}^n$ (fixed $n \in \mathbb{N}$) s.t for the set (E) of all discontinuities of f, $E \subseteq \cup_{i=1}^n B_{\delta_i}(p_i)$ with $\sum_{i=1}^n \delta_i < \varepsilon'$. So by extension, we can cover E by finitely many disjoint, closed intervals $\{I_j\}_{j=1}^n$ s.t $[u_j, v_j] = I_j$ ($u_j, v_j \in [a,b]$) and $\sum_{j=-1}^n v_j - u_j < \varepsilon'$. Per Rudin, we will keep track as the endpoints of these intervals as the primary objects of interest.

Each interval (u_j, v_j) is open in $[a,b]$. So the compliment of the union of these intervals with respect to $[a,b]$ is closed. Therefore, $A = [a,b] \setminus \cup_{j=1}^n (u_j, v_j)$ is closed thus compact since its bounded. So since f is continuous on A (compact), its also uniformly continuous on A. Therefore, $\exists \delta > 0$ s.t for $|s - t| < \delta$ $(s, t \in A)$, $|f(s) - f(t)| < \sigma$. This implies given $x_{i-1}, x_i \in A$ s.t $\Delta x_i < \delta$, $M_i - m_i \leq \sigma$. To see this, note that since σ is independent (not affected by the infimum)

$$|f(s) - f(t)| < \sigma \implies f(t) - \sigma < f(s) < f(t) + \sigma \implies \inf_{t \in B_\delta(s)} f(t) - \sigma \leq f(s) < \inf_{t \in B_\delta(s)} f(t) + \sigma$$

[2]We use the standard Rudin definition of M_i, m_i here and for the rest of the assignment

because otherwise $\inf_{B_\delta(s)} f(t)$ isn't a greatest lower bound; we can show a similar result for $\sup_{s\in B_\delta(t)} f(s)$.

Now form a partition $P = \{a = x_0 < x_1, \cdots < x_N = b\}$ of $[a, b]$ s.t each $u_j, v_j \in P$, given j, i s.t $x_{i-1} = u_j$, $x_i = v_j$, $P \cap (u_j, v_j) = \emptyset$ $(\forall j)$, and if x_{i-1} is not u_j $(\forall j)$ then $\Delta x_i < \delta$. Let $U = \{u_j\}_{i=1}^n$. So note that $\{x_i \in P | x_i \notin U\}$ can be expressed as a finite collection $\{y_i\}_{i=1}^M$, and the sum $\sum_{i=1}^M \Delta y_i$ telescopes to something positive but bounded by $b - a$. Now we have

$$\sum_{i=1}^N (M_i - m_i)\Delta x_i = \sum_{x_{i-1}\notin U} (M_i - m_i)\Delta x_i + \sum_{x_{i-1}\in U} (M_i - m_i)(\Delta x_i) \le \sigma \sum_{i=1}^M \Delta y_i + 2L \sum_{x_{i-1}\in U} (\Delta x_i) \le \varepsilon$$

since $\sum_{x_{i-1}\in U} (\Delta x_i) = \sum_{j=-1}^n v_j - u_j < \varepsilon'$. $U(P, f) - L(P, f) = \sum_{i=1}^N (M_i - m_i)\Delta x_i$, so $f \in \mathcal{R}$

(b) The same is not true for $\mathcal{R}(\alpha)$ with α increasing, but is for α continuously differentiable
For α monotonic, the part we want to exploit, to get a contradiction, that is different from the proof of part a is $\sum_{x_{i-1}\in U} (\Delta x_i)$. We can make a construction $\sum_{x_{i-1}\in U} (\Delta \alpha_i)$ is extremely large because of a "fat tail"; our only requirement is that its increasing. Because we could have countably infinite discontinuities, this blows up the term. An explicit counter-example: consider $\alpha(x) = .5\big(1 + \text{sgn}(x)\big)$ with f discontinuous at 0. No matter how small we shrink the distance between x, the difference in α about 0 will be 1, and the discontinuity will blow the sums up. For α continuously differentiable, note that implicitly we assume α is also monotonic from the definitions in class, so we invoke 6.17. If this is over-assuming (for monotonicity), then note the Jordan Decomposition, which uses monotonic α-type objects as inputs, which still gives us what we want.

4: α incr on $[a, b]$, α cts at $x_0 \in [a, b]$, $f(x_0) = 1, \& f(x) = 0$ if $x \ne x_0$. $f \in \mathcal{R}(\alpha), \int f d\alpha = 0$.

Fix $\varepsilon > 0$. By the continuity of α, pick $\delta > 0$ s.t $|x_0 - x| < \delta \implies |\alpha(x_0) - \alpha(x)|$. Therefore, we know that a partition $P = \{a = p_0 < p_1 < \cdots < p_n = b\}$ s.t $\delta p_i < .5\delta$ is well-defined. We know for some j, $x_0 \in [p_{j-1}, p_{j+1}]$. Further, that a sup of $f(x)$ over any interval that contains x_0 will be 1 and otherwise will be 0 (by assumption). For each i, pick $t_i \in [p_{i-1}, p_i]$, so that $\{t_i\}_{i=1}^n$ with P yields a tagged partition. Therefore (the LHS gives us the definition of $\mathcal{R}(\alpha)$ and $\int f d\alpha$)

$$\left|\sum_{i=1}^n f(t_i)\Delta\alpha_i\right| \le |f(t_j)|(\alpha(t_j) - \alpha(t_{j-1})) + |f(t_{j+1})|(\alpha(t_{j+1}) - \alpha(t_j)) \le \alpha(t_{j+1} - \alpha(t_{j-1}) < \varepsilon$$

5: $f \ge 0, f$ its on $[a, b]$, and $\int_a^b f(x)dc = 0$. $f(x) = 0\ \forall x \in [a, b]$

Assume, for a contradiction, that $\exists x_0 \in [a, b]$ s.t $f(x_0) > 0$. Because f is continuous, we know from HW10 # 5 that there must exists an interval that f is non-zero on (i.e there must be infinitely many points where $f(x) > 0$), otherwise we will arrive at a contradiction. Therefore, for each $\varepsilon > 0$, $\exists\delta > 0$ s.t for $|x - x_0| < \delta$ $(x \in [a, b]), |f(x) - f(x_0)| < \varepsilon$ and $|f(x)| > 0$. Now consider some $\delta_0 \in (0, \delta)$. It follows that f is non-zero on $[x_0 - \delta_0, x_0 + \delta_0]$, and because this interval is closed, the inf of f over this interval is also non-zero. Thus, for $P = \{a, x_0 - \delta_0, x_0 + \delta_0, b\}$ a partition of $[a, b]$, $L(P, f) > 0$ because $m_2 > 0$ by construction. So $\int_a^b f(x) \ge L(P, f) > 0$, contradiction.

6: Define functions $\beta_1, \beta_2, \beta_3$ s.t $\beta_1(0) = 0, \beta_2(0), \beta_3(0) = .5, \beta_j(x) = 1$ if $x > 0$ and $\beta_j = 0$ otherwise. f bounded on $[-1, 1]$

Credit to Marc for the notation and partition construction for the whole problem

(a) $f \in \mathcal{R}(\beta_1)$ iff $f(0+) = f(0)$. $\int f d\beta_1 = f(0)$
If $f(0+) = f(0)$, construct $P = \{x_0 = -1 < x_1 = 0 < x_2 < x_3 = 1\}$ a partition of $[-1, 1]$ ($x_2 \in [0, 1]$, otherwise fixed). Now let $M = \sup\limits_{x \in [x_2, x_3]} f(x)$ and $m = \inf\limits_{x \in [x_2, x_3]} f(x)$. β_1 will only be non-zero on the $(0, 1)$ interval (and will more precisely be one), so $[U - L](P, f, \beta_1) = M - m$ because there is only one such interval in our partition. By $f(0+) = f(0)$, as $x \to 0^+$, $M, m \to 0$, so $f \in \mathcal{R}(\beta_1)$ and the integral is exactly $f(0)$.

Fix $\varepsilon > 0$. If $f \in \mathcal{R}(\beta_1)$, we know there exists $P = \{x_0 < x_1 < \cdots < x_n\}$, a partition of $[-1, 1]$ s.t $[U - L](P, f, \beta_1) < \varepsilon$. For some j, $0 \in [x_{j-1}, x_j]$. By the properties established in the other direction, $[U - L](P, f, \beta) = M_j - m_j < \varepsilon$. Now choose $\delta > 0$ s.t $B_\delta(0) \subseteq [x_{j-1}, x_j]$. Then $|f(x) - f(0) \leq M_j - m_j < \varepsilon$ for $x \in B_\delta(0) \cap \mathbb{R}^+$. Therefore by definition, $f(0+) = \lim\limits_{x \to 0^+} f(x) = f(0)$ because our choice of δ can be arbitrarily small. So we once again have that the relevant integral is exactly $f(0)$ by definition

In both cases, we saw that $\int f d\beta_1 = f(0)$ because the difference between lower/upper sums is arbitrarily small, but the upper/lower sums themselves are $f(0)$

(b) a) holds for β_2
For this part, we will slightly modify the definitions and then rely upon the rigor we used in part a. Use instead $P = \{x_0 = -1 < x_1 < x_2 = 0 < x_3 = 1\}$ a partition of $[-1, 1]$, using $M = \sup\limits_{x \in [x_1, x_2]} f(x)$ and $m = \inf\limits_{x \in [x_1, x_2]} f(x)$. The only difference between a and b for this part now (other than WLOG-type notational changes, like using $\mathbb{R}^-$ instead) is that we have $\beta_2 \to 1$ (instead of $\beta_1 = 1$). So we just replace the equalities (where necessary) in the corresponding direction in part a with limits, and then from arguments we've done to death in previous homeworks this means that the upper and lower sums are converging to $f(0)$ as $x \to 0^-$ for the first direction, and for the second we still have a shrinking radius that will lead to $f(0)$.

(c) $f \in \mathcal{R}(\beta_3)$ iff f its at 0
From the above to cases, it follows that the difference of the upper and lower sums is $.5(M_j + M_{j-1} - m_j - m_{j-1})$ for some j. Then we can construct a partition that bounds this difference by $.5\varepsilon$ and for any arbitrary $\delta > 0$, $0 \in [x_{j-1}, x_{j+1}]$ with $x_{j+1} - x_{j-1} < \delta$. Then we can use the fact that $M_j + M_{j-1} - m_j - m_{j-1}) < \varepsilon$ to bound $|f(x) - f(0)|$, which gives continuity by definition. Analagous argumentation to the previous parts shows that $\in f d\beta_3 = f(0)$

(d) f its at 0 $\implies \int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0)$
This is simply a combination of the parts above

7: $f:(0,1]\to\mathbb{R}$ and $f\in\mathcal{R}$ on $[c,1]$ $(c>0)$. Let $\int_0^1 f(x)dx = \lim\limits_{c\to 0}\int_c^1 f(x)dx$ if limit is finite

(a) $f\in\mathcal{R}$ on $[0,1]$, integral definitions agree
By Rudin 6.20, because $f\in\mathcal{R}$ on $[a,b]$, $\lim_{c\to 0}\int_0^c f(x)dx = \lim_{c\to 0}F(c) = 0$ by the continuity of $F(\cdot)$. With clarification from Dr. Stokols, we can write $\lim a-b = \lim a - \lim b$ so long as each limit exists. Therefore by Rudin 6.12c

$$\lim_{c\to 0}\int_c^1 f(x)dx = \lim_{c\to 0}\int_0^1 f(x)dx - \lim_{c\to 0}\int_0^c f(x)dx = \int_0^1 f(x)dx$$

(b) Construct an f s.t the limit exists, but doesn't for $|f|$

Consider, fom HW9 # 1, $f(x)=\sin(1/x)/x$. Let $u=1/x$ $(du = x^{-2}dx$, so $dx = u^{-2}du)$. Then

$$\int_a^1 f(x)dx = \int_1^c u\sin(u)u^{-2}du = \int_1^c \sin(u)u^{-1}du$$

We make a change to the integration bounds because, analogous to part a, we want to consider a situation where the lower (integrating) bound is approaching 0 for $f(x)$, so then for $f(1/x)$ we consider an upper bound approaching ∞ (we will make the limiting arguments more precise at the end). Thus by Rudin 6.22 (with $F=u^{-1}$ and $g=\sin(u)$)

$$\int_1^c \sin(u)u^{-1}du = \frac{-\cos(u)}{u}\bigg|_1^c - \int_1^c \frac{-cos(u)}{u^2}du$$

Notice that $-u^{-1}\le \frac{-\cos(u)}{u}\le u^{-1}$ and $|\int_1^c \frac{-cos(u)}{u^2}du| \le \int_1^c \frac{1}{u^2} = \frac{-1}{u}\bigg|_1^c$. So as $c\to\infty$, $\frac{-\cos(u)}{u}\bigg|_1^c - \int_1^c \frac{-cos(u)}{u^2}du \to r\in\mathbb{R}$ by the squeeze theorem. Now, putting this all together

$$\lim_{a\to 0} f(x)dx = \lim_{c\to\infty}\int_1^c \sin(u)u^{-1}du = \lim_{c\to\infty}\frac{-\cos(u)}{u}\bigg|_1^c - \int_1^c \frac{-cos(u)}{u^2}du \to r$$

However, the same clearly does not hold for $|f|$. We have essentially established in class why this does not hold but we also offer the following. Consider an interval arbitrarily close to 0 and arbitrarily small. More precisely, $\exists\delta>0$ s.t $\forall\delta'\in(0,\delta)$, $|f(x)|$ cannot be bounded from above (on $(x-\delta', x+\delta')$ but its inf will always be 0 because the function is oscillating so quickly. In other words, $M_i - m_i$ will always take the form $\infty - 0$ no matter how small we make δ'. Therefore, $|f|\notin\mathcal{R}$.

Weekly Homework 12

Paul B.
Math 531: Real Analysis I

April 28, 2022

1: Let f_n be functions in $\mathcal{F}([a,b],\mathbb{R})$ and suppose $f_n \to f$

$[f]_{\text{Lip}}$ is the smallest positive constant C such that f is Lipschitz continuous (with C). Criticically for part a), this implies if $\exists C$ s.t $|f(x)-f(y)| \le C|x-y|$ $(x \ne y)$, then $[f]_{\text{Lip}} \le C$

(a) $[f]_{\text{Lip}} \le \sup_n [f_n]_{\text{Lip}}$
If $\sup_n [f_n]_{\text{Lip}} = \infty$, the inequality holds trivially (nothing can be $> \infty$). So assume $\sup_n [f_n]_{\text{Lip}} \ne \infty$. So all f_n are Lipschitz and $\exists C \in \mathbb{R}^+$ s.t $|f_n(x) - f_n(y)| \le C|x-y|$ $\forall\, n$ and $x, y \in [a,b]$ $(x \ne y)$.

Fix $\varepsilon > 0$ and let $x, y \in [a,b]$. By $f_n \to f$, $\exists N_1$ s.t $\forall n > N_1 |f(x) - f_n(x)| < \varepsilon$. Also, $\exists N_2$ s.t $\forall n > N_2 |f(y) - f_n(y)| < \varepsilon$. Take $N = \max\{N_1, N_2\}$. Then by the triangle inequality

$$|f(x)-f(y)| \le |f(x)-f_n(x)|+|f(y)-f_n(y)|+|f_n(x)-f_n(y)| < 2\varepsilon+\sup_n |f_n(x)-f_n(y)| = 2\varepsilon+C\cdot|x-y|$$

for all $n > N$. Because a strict equality holds for all ε and ε is arbitrary,from past HW we know we can take out ε and use a weak inequality, yielding $|f(x)-f(y)| \le C|x-y|$, so $[f]_{\text{Lip}} \le \sup_n [f_n]_{\text{Lip}}$.

(b) Construct $f_n \to f$ s.t $\sup_n \sup_{(a,b)} |f_n'(x)| < \infty$ but f not differentiable
Consider[1] $f_n(x) = \sqrt{x^2 + n^{-1}}$. Because of results we've shown on past howeworks, it should be clear that $f_n(x) \to \sqrt{x^2} = |x|$, which is not differentiable at 0.

To complete the proof, we will show $\sup_n \sup_{(-2,2)} |f_n'(x)| = 1$. $f_n'(x) = \frac{x}{\sqrt{x^2+n^{-1}}}$. First we will show: on $[-2,2]$, $f_n(x) \in [-1,1] \forall n$. One might have a concern of what happens at/near 0, but the behavior is normal even in the limit. To see this, fix n and let $\varepsilon \in (\frac{1}{2n}, \frac{1}{n})$ (so clearly $\varepsilon \to 0$). We can comfortably bound $f_n'(\varepsilon)$ (similar arguments for $f_n'(-\varepsilon)$ also) above and below by

$$0 < \frac{1}{2\sqrt{n}} < \frac{\frac{1}{2n}}{\sqrt{n^{-2}+n^{-1}}} < \frac{\varepsilon}{\sqrt{\varepsilon^2+n^{-1}}} < f_n'(\varepsilon) < \frac{n^{-1}}{\sqrt{\varepsilon^2+n^{-1}}} < \sqrt{n^{-1}} \le 1$$

Further, $x \to 1, n \to \infty \implies f_n'(x) \to 1$ and $x \to -1, n \to \infty \implies f_n'(x) \to -1$ (clear from the form of $f_n'(x)$). Further, $f_n'(x)$ is an increasing function on $\mathbb{R}$ (this is also clear, consider Rudin 5.13). So we have proved that $f_n(x) \in [-1,1]$ for $x \in [-2,2]$ and for all n, and that $\sup_n \sup_{(-2,2)} f_n' = -\inf_n \inf_{(-2,2)} f_n' = 1$.

So $\sup_n \sup_{(-2,2)} |f_n'(x)| = 1$, but $f_n \to f$ not differentiable at 0.

[1]From *Real Mathematical Analysis* by Pew, p. 220. Also, Dr. Stokols said f just has to not be diff at one point

2: For (f_n) bounded s.t $f_n \rightrightarrows f$, (f_n) are uniformly bounded

Since[2] $f_n \rightrightarrows f$, $\exists N$ s.t $\forall n > N$ $|f_n(x) - f(x)| < 1$ $(\forall x)$. So $|f_n(x)| \leq |f(x)| + 1$. Since each f_n bounded, given n, $\exists M_n \in \mathbb{R}$ s.t $|f_n(x)| \leq M_n$ $(\forall x)$. Thus f is bounded; define M_0 s.t $|f(x)| < M_0$. Recall the previous N. Let $M = \max\{M_1, \ldots, M_N, M_0 + 1\}$. Then $|f_n(x)| < M$ $\forall x, n$.

3: Consider $f_n, g_n \rightrightarrows f, g$ on $E \subset \mathbb{R}$

(a) $f_n + g_n \rightrightarrows f + g$ on E
Fix $\varepsilon > 0$. By $f_n \rightrightarrows f, g_n \rightrightarrows g$, (using the shortcut established in #1) $\exists N$ s.t $\forall n > N$

$$|f_n(x) - f(x)| < .5\varepsilon \ \underline{\textbf{and}}\ |g_n(x) - g(x)| < .5\varepsilon$$

for any x. By the triangle inequality

$$\implies |[f_n + g_n](x) - [f + g](x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon$$

(b) For f_n, g_n bounded, $f_n \cdot g_n \rightrightarrows f \cdot g$ on E

By #2, f_n, g_n are uniformly bounded. So $\exists M$ s.t $|f_n(x)|, |g_n(x)| \leq M$ for any n, x. Also from work we did for #2, we similarly have $|f(x)|, |g(x)| \leq M$. Then by the triangle inequality

$$|[f_n \cdot g_n](x) - [f \cdot g](x)| = |[f_n \cdot g_n](x) - [g_n \cdot f](x) + [g_n \cdot f](x) - [f \cdot g](x)| \leq |g_n(x)||f_n(x) - f(x)| + |f(x)||g_n(x) - g(x)|$$

Fix $\varepsilon > 0$. Using the above result and $|g_n|, |f| < M$, per part a) $\exists N$ s.t $\forall n > N$ and any x

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2M} \ \underline{\textbf{and}}\ |g_n(x) - g(x)| < \frac{\varepsilon}{2M} \implies |[f_n \cdot g_n](x) - [f \cdot g](x)| < \varepsilon$$

4: Consider $f_n, g_n \rightrightarrows f, g$ on $\mathbb{R}$, but $f_n \cdot g_n \not\rightrightarrows f \cdot g$

Let[3] $f_n(x) = g_n(x) = x^2 + n^{-1}$ and $f(x) = g(x) = x^2$. If we fix $\varepsilon = N^{-1}$ (any $N \in \mathbb{N}$) then for $n > N$

$$|f_n(x) - f(x)| = |g_n(x) - g(x)| = |n^{-1}| < \varepsilon$$

for any x. So clearly, f_n, g_n converge using a definition of N that does not depend on x, meaning $f_n, g_n \rightrightarrows f, g$. However

$$[f_n \cdot g_n](x) = (x + n^{-1})^2 = x^2 + 2xn^{-1} + n^{-2} \implies |[f_n \cdot g_n](x) - [f \cdot g](x)| = |2xn^{-1} + n^{-2}|$$

For x fixed, $|2xn^{-1} + n^{-2}| \to 0$, so $f_n \cdot g_n \to f \cdot g$ pointwise. However, how large this term is depends on the value of x, and there is no way to eliminate the x because the problem is as simplified as it can get. So per examples in class, there is no way to fix N solely based on ε, therefore there is no uniform convergence. Here's a bit more of a formal argument to that end in case one feels it is necessary. Take N_ε (N fixed soley based on ε). Then for $n > N_\varepsilon$, we can simply consider $|2x| > n$ and $|2xn^{-1} + n^{-2}| > 1$, so will not be less than an arbitrary ε.

[2] For this whole HW, "for any" or "$\forall$" x means for any $x \in \mathbb{R}$ (per Dr. Stokols). Also $\rightrightarrows$ denotes uniform convergence
[3] Credit to Angel for this counterexample

5: Consider $f(x) = \sum(1+n^2x)^{-1}$

Define[4] $f_n(x) = \frac{1}{1+n^2x}$

(a) For what values does the series converge absolutely
At $x = 0$, the series $(\sum 1)$ diverges. But if we fix $x > 0$, $0 < \sum \frac{1}{1+n^2x} < \sum \frac{1}{n^2x}$ converges by the comparison test. Now consider $x < 0$. We cannot construct the same bound because if $x = \frac{1}{n^2}$ for some $n \in \mathbb{N}$ then we have a 0 in the denominator for the nth term in the series, so the series itself is undefined. However, if no such n exists we have a similar limit comparison test argument of $-\infty < \sum \frac{1}{n^2x} < \sum \frac{1}{1+n^2x} < 0$. Therefore, define the set $C = \{0\} \cup \{x \in \mathbb{R} |\ \exists n \in \mathbb{N} \text{ s.t } x = \frac{1}{n^2}\}$. Then we have shown if $x \in \mathbb{R}\backslash C$, the series converges absolutely.

(b) For what values does the series converge uniformly
Trivially, $1 + xn^2$ is increasing in x, so $\frac{1}{1+xn^2}$ is decreasing in x. Thus, fix $c > 0$. Then we have $\sup\limits_{(c,\infty)} |f_n(x)| = \frac{1}{1+cn^2}$. So by part a), $\sum \sup\limits_{(c,\infty)} |f_n(x)|$ converges. So for $x \in (c, \infty)$ (recall c is an arbitrary positive number), $f_n(x) \leq \sup\limits_{(c,\infty)} |f_n(x)|$ and by Rudin 7.10 $\sum f_n(x)$ converges uniformly. We have a similar result (of uniform convergence by 7.10) for $c < 0$ and $\sum \sup\limits_{(-\infty,c)} |f_n(x)|$, with the caveat we established in part a). Thus, $\sum f_n(x)$ converges uniformly if $x \in \mathbb{R}\backslash C$ (doesn't if $x \in C$).

(c) Is f continuous wherever the series converges
Given $x \in \mathbb{R}\backslash C$, we have shown in class and on analogous HW problems that $f_n(x)$ is continuous. Therefore, $\sum f_n(x)$ is a sum of continuous functions and is thus continuous. We have also showed that $\sum f_n(x) \rightrightarrows f(x)$, so by Rudin 7.12 f is continuous on $\mathbb{R}\backslash C$, the set where the series converges.

(d) Is f bounded
f is not bounded on $\mathbb{R}$. For instance, take our example of $x = 0$ from part a), which is unbounded.

6: For $I(x) = 0$ if $x \leq 0$ and 1 otherwise, if $(x_n) \in (a, b)$ distinct sequence and $\sum |c_n|$ converges, $S_n(x) = \sum_{i=1}^n c_i I(x - x_n)$ converges uniformly, and f is continuous for all $x_n \neq x$

Note the definition of the partial sums, $S_n(x)$, above (per conversation with Dr. Stokols). Further $|S_n(x)| = |\sum_{i=1}^n c_i I(x - x_n)| \leq \sum_{i=1}^n |c_n| \cdot 1$, and the partials sums of $\sum |c_n|$ converge since its a convergent series. Therefore, we can apply Rudin 7.10 and say S_n converges uniformly. Further, by a property of a convergent sums it converges to a constant (for each x not equal to a distinct sequence term), so $f(x) \in \mathbb{R}$ and is thus continuous. Here is some more detail: because we assume that (x_n) distinct sequence terms and $x \neq x$, S_n is continuous. To see why, consider that $I(x - x_n)$ will never be 0, so given $x \in \mathbb{R}\ \exists y \in \mathbb{R}$ s.t $|x - y| < \delta$ (δ arbitrarily small) and $y \neq x_n$ $\implies I(x - x_n) = I(y - x_n)$. This means S_n continuous, so apply Rudin 7.12 to see that since we established also $S_n \rightrightarrows f$, f must be continuous.

[4]Note $\sum x_n$ denotes an infinite sum, per Rudin. Credit to Marc for the construction of C.

Weekly Homework 13

Paul B.
Math 531: Real Analysis I

April 28, 2022

1: Given $I = [-A, A] \subseteq \mathbb{R}$, there is a sequence of polynomials that converge uniformly to $|x|$ on I

For clarity of what we are trying to prove, we must show that given $x \in I$, there exists $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ s.t $a_i \in \mathbb{R}\,(\forall i)$ and $P_n \rightrightarrows |x|$. We will do this by first showing a general uniform convergence result, then showing that $e^x \rightrightarrows \sum x^n/n!$ and $\int \operatorname{erf}(x/\delta) \rightrightarrows \int \operatorname{sgn}(x)$ $(\delta \to 0)$, with lots of substeps. Finally, note Dr. Stokols misspecified the constants a bit in the writeup of the problem, and I will be using $C = .5\sqrt{\pi}$ in relevant places to remedy this.

First, note the following property of uniform convergence I confirmed with Dr. Stokols in office hours. Assume that $\sum_{n=1}^{N} f_n \rightrightarrows \sum f_n$ $(N \to \infty)$. Then

$$\sum \int f_n = \lim_N \sum_{n=1}^{N} \int f_n = \lim_N \int \left(\sum_{n=1}^{N} f_n \right) = \int \sum f_n \tag{1}$$

We will use that uniform convergence implies $\sum \int f_n = \int \sum f_n$ twice in this proof.

For $[-R, R] \subseteq \mathbb{R}$, $\sum R^n/n!$ converges (ratio test - $\limsup a_{n+1}/a_n = \limsup R/(n+1) = 0$). Further, given $z \in [-R, R]$, $|z^n/n!| \leq R^n/n!$. So by Rudin 7.10 $\sum z^n/n!$ converges uniformly. This is the Taylor series for e^z, so $\sum z^n/n! \rightrightarrows e^z$. By definition and (1) above

$$C \cdot \operatorname{erf}(z) = \int_0^z e^{-t^2} dt = \int_0^z \left(\sum \frac{(-1)^n t^{2n}}{n!} \right) dt = \sum \frac{(-1)^n t^{2n+1}}{(2n+1)n!} \Bigg|_0^z = \sum \frac{(-1)^n z^{2n+1}}{(2n+1)n!}$$

So now we have shown that if we fix a closed interval in $\mathbb{R}$, we can show uniform convergence. We will exploit this fact by creating intervals that correspond to an input that is going towards ∞ (which guarantees both that the interval is large enough and that we still have uniform convergence). Fix $\varepsilon > 0$. We will think of this as how "good" we want our approximation to be. Now consider a sequence $(\delta_k) \to 0$ s.t $\delta_k > 0\,(\forall k)$. Further, fix $x \in I = [-A, A]$, our given interval. Then define $A_k \in \mathbb{R}^+$ s.t $x/\delta_k \in [-A_k, A_k]$. Now we have done the work to invoke the uniform convergence argument above and say that for $\exists N_k^1$ s.t $|\operatorname{erf}(z) - \sum_{n=0}^{N_k^1} \frac{(-1)^n z^{2n+1}}{(2n+1)n!}| < \frac{\varepsilon}{4}$ for any $z \in [-A_k, A_k]$. Note that this argument still holds if we replace N_k^1 with any $M > N_k^1$. Now as $k \to \infty$ (equivalently as $\delta_k \to 0$), we are given that $C \cdot \operatorname{erf}(x/\delta_k) \to C \cdot \operatorname{sgn}(x)$. Therefore, given ε and δ_k, $\exists N_k^2$ s.t $\forall i > N_k^2$ $|\operatorname{erf}(x/\delta_k) - \operatorname{sgn}(x)| < \frac{\varepsilon}{4}$. This implies that for $N_k = N_k^1 + N_k^2$, by the triangle inequality arguments we've used throughout this semester $\left| \sum_{n=0}^{N_k} \frac{(-1)^n (t/\delta_k)^{2n+1}}{(2n+1)n!} - \operatorname{sgn}(t) \right| < .5\varepsilon$.

The problem is the previous argument (with N_k^2) is that we don't have uniform convergence. We need the uniform convergence of certain integrals. We will achieve this by showing convergence over the entire interval using Darboux sums, giving us a bound to once again invoke the M-test. Let $f_{N_k}(x) = \sum_{n=0}^{N_k} \frac{(-1)^n (x/\delta_k)^{2n+1}}{C(2n+1)n!} - \text{sgn}(x)$. We WTS

$$(\text{as } k \to \infty) \int_0^x \text{erf}(t/\delta_k)dt - \int_0^x \text{sgn}(t)dt = \int_0^x \Big(\text{erf}(t/\delta_k) - \text{sgn}(t)\Big)dt \rightrightarrows 0$$

Or equivalently by uniform convergence established for erf

$$(\text{as } k \to \infty) \int_0^x f_{N_k}(t)dt = \int_0^x \left(\sum_{n=0}^{N_k} \frac{(-1)^n (t/\delta_k)^{2n+1}}{C(2n+1)n!} - \text{sgn}(t)\right) dt = \int_0^x \sum_{n=0}^{N_k} \frac{(-1)^n (t/\delta_k)^{2n+1}}{C(2n+1)n!} dt - \int_0^x \text{sgn}(t)dt \rightrightarrows 0$$

Note that $\text{erf}(x) \leq \text{sgn}(x)$ for $x > 0$, and likewise $f_{N_k}(x) \leq 0$ since the partial sum approximation is bounded by $\text{erf}(x)$ from above. Further, the difference between $\text{erf}(x)$ and $\text{sgn}(x) \to 1$ as $x \to 0$, so because these functions are symmetric (and $f_{N_k}(x) \geq 0$ for $x < 0$ by similar arguments above), this implies that $f_{N_k}(x) \to -.5$ for $x \to 0^+$ and $f_{N_{k}(x)} \to .5$ as $x \to 0^-$. We will use this fact when making sense of the upper/lower sums, noticing implicitly from the triangle inequality argument that as k increases our approximation becomes better. WLOG take $x > 0$ and recall the usual $m_i = \inf_{x \in [x_{i-1}, x_i]} f_{N_k}(x)$ notation for Darboux sums (same for M_i). By the previous convergence argument and the properties of $f_{N_k}(x)$ for $x > 0$, there exists a K s.t $\forall k > K$ (which generates a N_k making our partial sums an arbitrarily good approximation of $\text{sgn}(x)$) we can create a partition of the interval $[0, A]$ will yield $m_1 = -.5$ and $-.5\varepsilon < m_i \leq 0$ for all $i > 1$. Again, because we can make our approximation arbitrarily good by taking i large (shrinking δ_k), this also means the Δ_i needed to make such a partition will also be shrinking. Thus, the aforementioned partition would have the property that $-.5\varepsilon < M_i \leq 0$ $(i \in \mathbb{N})$ since $f_{N_k} \to 0$ quickly for large k (our partial sums are becoming arbitrarily close to $\text{sgn}(x)$). Now we have established the relevant properties to make a well-defined construction of a partition that gives us the results that we want. Define a partition of $[0, A]$ $P_k = \{x_0 = 0 < x_1 < \cdots < x_{n_k} = A\}$ such that $m_1 = -.5$ and $|M_1|, |m_i|, |M_i|, < .5\varepsilon$ for all $i > 1$ and has uniform distance between partition points (i.e. for some $a_k \in \mathbb{R}^+$, a sharp[1] constant, $\Delta_i = a_i$ $\forall i$). Now notice from how we have defined this partition, as k grows, and thus δ_k shrinks, the number of partition points (n_k) grows, so the partition becomes finer and the uniform distance between partition points shrinks (in other words $a_k \to 0$). Therefore we have

$$[U - L](f_{N_k}, P_k) = \sum_{i=1}^{n_k} (M_i - m_i)\Delta_i = (M_1 - m_1)a_k + \sum_{i=2}^{n_k} (M_i - m_i)\Delta_i < .5a_k + \varepsilon \sum_{k=2}^{n} \Delta_i < .5a_k + \varepsilon \cdot A$$

Thus $[U - L](f_{N_k}, P_k) \to A\varepsilon$ and therefore f_{N_k} is integrable on $[0, A]$. We can easily modify the above argument (by removing the relevant terms) to show that the upper and lower sums themselves are 0 on $[0, A]$. Further, consider that because $[0, x] \subseteq [0, A]$, we have that the absolute value of the upper and lower sums of $\int_0^x f_{N_k}(t)$ are weakly bounded by the upper and lower sums of $\int_0^A f_{N_k}(t)$, which we just developed and showed converge. Thus[2], by Rudin 7.10 $\int_0^x f_{N_k}(t) \rightrightarrows 0$,and using the decomposition at the top of the page $\int_0^x \sum_{n=0}^{N_k} \frac{(-1)^n (t/\delta_k)^{2n+1}}{C(2n+1)n!} dt \rightrightarrows \int_0^x \text{sgn}(t)dt$. We can get a very similar result on $[-A, 0]$ (considering $x < 0$) by making slight adjustments to the proof, for instance we would have $M_1 = .5$ in this direction. Therefore, we have shown uniform convergence on $[-A, A]$ using Darboux sums to foster a M-test argument.

[1]This is from class; it means that a_k is not bigger then it needs to be

[2]See appendix at the end of the proof for a rigorous bounding of the upper/lower sums to use the M-test

By (1) at the very beginning, we can consider $\int_0^x \sum_{n=0}^{N_k} \frac{(-1)^n(t/\delta_k)^{2n+1}}{C(2n+1)n!}dt = \sum_{n=0}^{N_k} \int_0^x \frac{(-1)^n(t/\delta_k)^{2n+1}}{C(2n+1)n!}dt$, even in the limit (as $k \to \infty$). We know that the integral of a polynomial is a polynomial. Therefore, given $x \in A$, we can construct a polynomial of order N_k (with $N_k + 1$ or less terms), denoted $P_{N_k}(x)$ with coefficients dependent on δ_k (i.e. we increase the order by increasing k, which subsequently changes all the coefficients because δ_k also changes), such that $P_N(x) \rightrightarrows \int_0^x \text{sgn}(t)dt = |x|$

Appendix - *formalizing the use of bounds through Darboux sums over the entire interval*
Recall that we have a partition P_k for $[0, A]$. We can take k large enough such that there always exists $j < k$ s.t P_j (defined the same way, so there are $n_j < n_k$ partition points) is a partition for $[0, x]$ on f_{N_k}. So now consider the terms of the upper sums $U(f_{N_k}, P_j) = \sum_{i=1}^{n_j} M_i\Delta_i$. Define $S_i^j = M_i\Delta_i$ for $i \in [1, n_j]$ and 0 otherwise. Define $S_i^k = M_i\Delta_i$ for $i \in [1, n_k]$. Therefore, $U(f_{N_k}, P_j) = \sum_{i=1}^{n_j} S_i^j$ and $|S_i^j| \le |S_i^k|$ by construction. We showed in the main body of the proof $\sum S_i^k$ is absolutely convergent because the terms are weakly negative. So now we can use Rudin 7.10 and say $\sum S_i^j$, and therefore $U(f_{N_k}, P_j)$, is uniformly convergent to 0. The argument to show the same for the lower sums follows nearly identically. Because the upper and lower sums uniformly converge to 0, the integral does as well.

2: (f_n) cont s.t $f_n \rightrightarrows f$ on E

(a) $x_n \to x$ $(x_n, x \in E) \implies \lim_n f_n(x_n) = f(x)$
Fix $\varepsilon > 0$. By uniform continuity, $\exists N_\varepsilon$ s.t $\forall n > N_\varepsilon$ $|f_n(x) - f(x)| < .5\varepsilon$ $(x \in E)$. Since this holds for any $x \in E$, it also must hold for any element of $(x_n) \in E$. By Rudin 7.12, since each f_n continuous and $f_n \rightrightarrows f$, f is continuous. So by continuity and $x_n \to x$, exists M_ε s.t $\forall n > M_\varepsilon$ $|f(x_n) - f(x)| < .5\varepsilon$. Combining results, by the triangle inequality, for all $n > (N_\varepsilon + M_\varepsilon)$

$$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon$$

(b) $\lim f_n(x_n) = f(x) \not\Longrightarrow x_n \to x$

Consider[3] $f_n(x) = x/n$ $x_n \in (0, 1)$ s.t $(x_n) \to 1$. Then $f_n(x_n) \rightrightarrows 0$. So $f(x) = 0$. But $f(0) = 0$, and $x_n \not\to 0$. So putting this all together, $\lim f_n(x_n) = 0 = f(0)$, but $x_n \not\to 0$.

3: Suppose g and f_n on $(0, \infty)$ are $\mathcal{R}$ on $[t, T]$ $(t, T \in \mathbb{R}^+)$, $|f_n| \le g$, $f_n \rightrightarrows f$ on every $K \subset \mathbb{R}^+$ compact and $\int_0^\infty g(x)dx < \infty$. Then $\lim_n \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$

For $[t, T] \subseteq \mathbb{R}^+$ (compact and arbitrary), since $|f_n| \le g$ (for each n) and $f_n \rightrightarrows f$, we know morally that since integration preserves ordering, $\exists M$ s.t $\int_t^T |f(x)|dx \le \int_t^{T+M} |g(x)|dx$ (otherwise g could not be a uniform bound on objects converging to f), or in other words $\int_t^T |f(x)|dx \le \int_t^\infty |g(x)|dx$. $\int_t^T |f(x)|dx$ is increasing in T due to the absolute value, so by the monotonic convergence theorm its convergent $(T \to \infty)$. By standard metric properties $|\int_t^T f(x)dx| \le \int_t^T |f(x)|dx$, so $\int_t^T f(x)dx$ must also be convergent. By given/established properties, this implies $\int_t^T f_n(x)dx$ is also converges.

Fix $\varepsilon > 0$. Since $\int_t^\infty |g(x)|dx < \infty$, we must have upper/lower bounds making the integral arbitrarily small (otherwise contradiction). In other words, $\exists a, b \in \mathbb{R}^+$ s.t

$$\int_0^a |g(x)|dx < \frac{\varepsilon}{6} \ \underline{\textbf{and}} \ \int_b^\infty |g(x)|dx < \frac{\varepsilon}{6}$$

[3] Credit to Marc for the counterexample. We showed uniform convergence of x/n earlier in the year

By $f_n \rightrightarrows f$ on $[a.b]$, $\exists N$ s.t for $n > N$ $|f_n(x) - f(x)| < \varepsilon(b-a)^{-1}$. Now taking $t \in [0, a]$

$$\left|\int_t^\infty f_n(x)dx - \int_t^\infty f(x)dx\right| \leq \int_t^a |f_n(x) - f(x)|dx + \int_a^b |f_n(x) - f(x)|dx + \int_b^\infty |f_n(x) - f(x)|dx$$
$$\leq 2\int_t^a g(x)dx + \frac{\varepsilon(b-a)}{3(b-a)} + 2\int_b^\infty g(x)dx$$
$$< \varepsilon$$

a set from $\varepsilon > 0$. So a, and therefore t, are arbitrarily small. Thus, $\lim_n \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$.

4: (f_n) incr on $\mathbb{R}$ s.t $f_n(x) \in [0, 1]$ w/ $f_{n_k} \to f$. Convergence uniform on compact sets if f cont

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We will follow[4] the $(i), (ii), (iii)$ strategy laid out in Rudin

Let $r_1 \in \mathbb{Q}$. By the properties of f_n, $f_n(r_1) \in [0, 1]$ for any n. By $[0, 1]$ compact and $(f_n(r_1)) \in [0, 1]$, there is a subsequence $(f_{n_{j_1}}(r_1)) \to f(r_1)$ by Bolzano-Weirstrass. Given $r_2 \in \mathbb{Q}\,(r_1 \neq r_2)$, we can similarly construct a subsequence $(f_{n_{j_2}}(r_2)) \to f(r_2)$. From Rudin Ch. 2 (Cardinality), we can construct a sequence (r_n) that contains every rational number. So continuing iteratively from before, we have $f_{n_k} = \{f_{n_{j_i}}\}_{i=1}^\infty$, so that $(f_{n_{jm}}(r_m)) \to f(r_m)$, so we have convergence to all rational points.

Let $f^*(x) = \sup\{f(r) | r \leq x, r \in \mathbb{Q}\}$ $(x \in \mathbb{R})$. Clearly for any $r \in \mathbb{Q}$, $f^*(r) \geq f(r)$. Assume that $f^*(r) \neq f(r)$, so $f^*(r) > f(r)$. Then $\exists p \in \mathbb{Q}\,\text{and}\, t \in \mathbb{R}$ s.t $t \in (f(r), f(p))$ with $p < r$ (since the sup is over $p \leq r$). By the initial definition, $f(p) = \lim f_{n_k}(p)$ and $f(r) = \lim f_{n_k}(r)$. So $\exists N_k$ s.t $\forall n_k > N_k$ $t \in (f_{n_k}(r), f_{n_k}(p))$, contradicting strictly increasing assumption. Thus, $f^*(r) = f(r)$.

Fix $\varepsilon > 0$. Let $K \subseteq \mathbb{R}$ compact. By continuity, for $c \in K$ $\exists\, \delta_c$ s.t $|f(x) - f(c)| < \frac{\varepsilon}{4}$ if $x \in B_{\delta_c}(c)$. By our limit definitions, there must exists a unifying N s.t for $k > N$

$$|f_{n_k}(c - \delta_c)| < .5\varepsilon \ \underline{\textbf{and}}\ |f_{n_k}(c + \delta_c)| < \frac{\varepsilon}{4}$$

So given $x \in B_{\delta_c}(c)$ and $k > N$, by the monotonicity of f_{n_k}

$$f(c) - .5\varepsilon < f(c - \delta_c) - .25\varepsilon < f_{n_k}(c - \delta_c) \leq f_{n_k}(x) \leq f_{n_k}(c + \delta_c) + .25\varepsilon < f(x) + .5\varepsilon$$

This implies, again given $x \in B_{\delta_c}(c)$ and $k > N$, $|f_{n_k}(x) - f(x)| < \varepsilon$ by the triangle inequality. Because we can take a union over $B_{\delta_c}(c)$ to cover K, there must exists $\{c_i\}_{i=1}^M$ $(M \in \mathbb{R})$ s.t. $K \subseteq \cup_{i=1}^M B_{\delta_i}(c_i)$. Let N_i be the relevant N (for each $i \in [1, M]$). Then take $N^* = \max_i N_i$. So for $k > N$ $|f_{n_k}(x) - f(x)| < \varepsilon$ $(x \in K)$. N independent of x, so $(f_{n_k}) \rightrightarrows f$ on any $K \subseteq \mathbb{R}$

5: f cont on $\mathbb{R}$, $f_n(t) = f(nt)$ and (f_n) equicont on $[0, 1]$. This implies f is constant on $[0, \infty]$

Fix $\varepsilon > 0$ and $x, y > 0$ s.t $x \neq y$. By the continuity of f, given $a, b \in [0, 1]$, $\exists \delta > 0$ s.t $|f(a) - f(b)| < \varepsilon$ for $a \in B_\delta(b)$. By the Archemedian property, $\exists N \in \mathbb{N}$ s.t we can let $a = x/N, b = y/N$ (we make them arbitrarily small, so they will be arbitrarily close together. By $f_n(t) = f(nt)$ and f_n equicontinuous on $[0, 1]$, for $n > N$

$$|f(x) - f(y)| = |f_n(x/n) - f_n(y/n)| < \varepsilon$$

since $d(x/n, y/n) \leq d(x/N, y/N)$. ε arbitrary, so $f(x) = f(y)$ for all $x \neq y$, thus constant.

[4] Credit to Greg for notation